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THEORY OF STOCHASTIC OPTIMAL TRACKING SYSTEMS

by

V. H. Syed

EE 7708

December 1977

SCHOOL OF ENGINEERING

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20. Abstract (continued)

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The systems in the open-loop as well as the closed-loop configurations are studied. For each case, the appropriate compensators are designed both in terms of their impulse response functions and in terms of their state-space realizations. Finally, the conditions for the stability of the resulting systems are derived.

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THEORY OF STOCHASTIC OPTIMAL TRACKING SYSTEMS

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V. H. Syed

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ABSTRACT

Theory of Stochastic Optimal Tracking Systems

by

Vaqar H. Syed

Doctor of Philosophy in Engineering

University of California, Irvine, 1977

Professor James S. Meditch, Dissertation Supervisor

Professor Allen R. Stubberud, Chairman

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CONTENTS

CONTENTS (continued)

LIST OF FIGURES

CHAPTER 1

INTRODUCTION

1.1 *GENERAL*

The problem of optimal tracking of random signals by linear plants is a basic ingredient of many communications and control system designs. A qualitative statement of the problem is as follows:

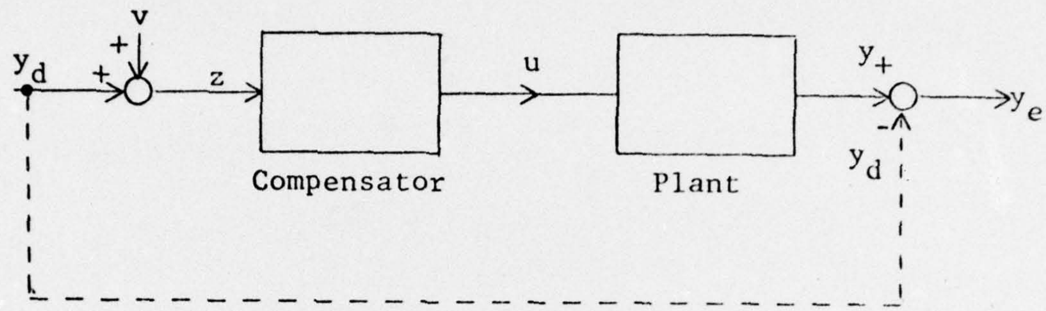
> Given a dynamic plant and a random signal corrupted by an additive noise as its input, how can one modify the plant dynamics such that the plant output follows the input in an optimal manner?

Obviously, this vague question must be quantified to obtain a tractable design problem. To accomplish this end, we make reasonable assumptions about the signal, the noise, and the plant, and define a suitable criterian for "optimality".

In this dissertation, we will consider only linear plants. The plant parameters are allowed to be time-varying, but are required to be continuous in the entire interval of interest. The random signal is assumed to be a stochastic process which can be generated by passing white noise through a finite-dimensional linear system with arbitrary initial conditions. The parameters of the system generating the stochastic process can also vary with time, but must also be continuous throughout the time

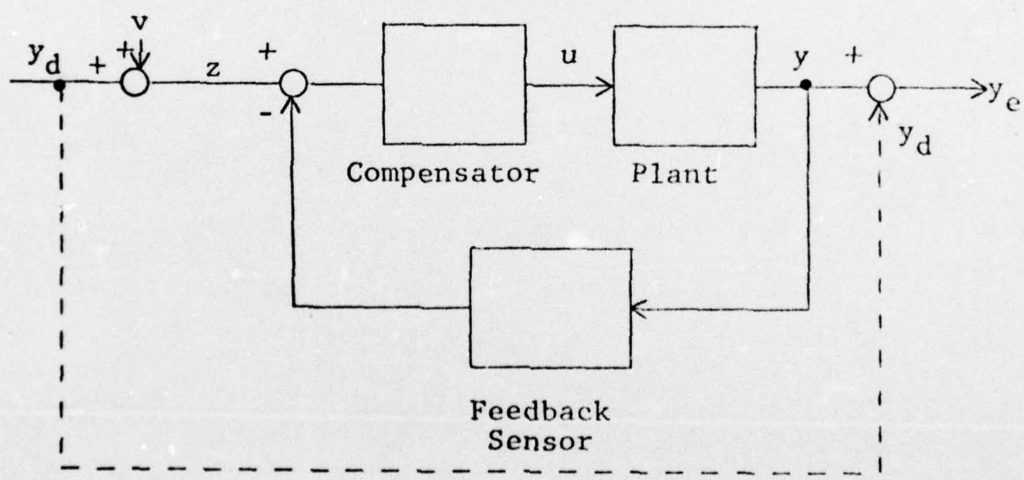
interval of interest. Note that the resulting process in general will be nonstationary. All noise processes are assumed to be white, although colored noise processes can be easily accommodated in this theory. The optimization criterion used here is such that the mean-square tracking error, defined as the difference between the plant output and the original random signal is minimized. Finally, since the modification of plant dynamics actually involves placing an appropriate compensator in the system, care will be taken to keep the mean-square value of the compensator output to within some specified bounds.

As pointed out above, the tracking problem requires the design of an appropriate compensator to modify the plant dynamics. The compensator can be placed in the system in either of the two configurations shown in Figure 1.1 and 1.2. The compensator as shown in Figure 1.1 will be called an "Open-loop Cascade Compensator", and the one shown in Figure 1.2 will be termed a "Closed-loop Cascade Compensator". The related design problems will be called "the Open-loop Problem" and the "Closed-loop Problem", respectively.



OPEN-LOOP PROBLEM

Figure 1.1

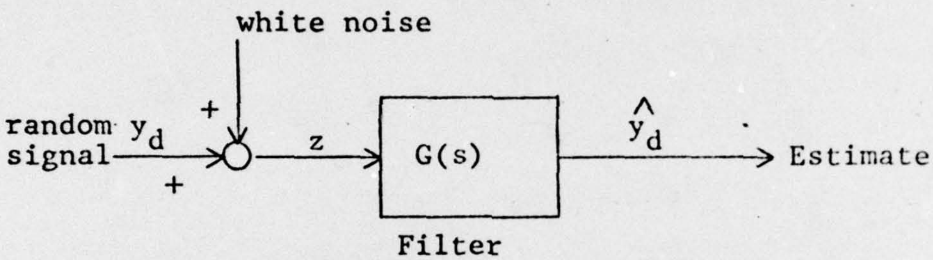


CLOSED-LOOP PROBLEM

Figure 1.2

1.2 HISTORICAL BACKGROUND

A special case of the above problem was studied by Wiener in the context of stochastic filter design [1]. His problem consisted of the design of an estimator (filter) whose output would optimally follow a random signal available in the presence of an additive white noise (see Figure 1.3).



WIENER FILTER

Figure 1.3

He modeled the random signal as a stationary stochastic process with rational power density spectrum and sought the transfer function of an optimal filter such that the mean-square error between the filter output and the given random signal is minimized.

Thus, if the "plant" in this work is a scalar identity system (impulse response $\delta(t-\tau)$), the random signal and the noise are stationary, and there is no constraint on the compensator output, the stochastic optimal tracking problem reduces to the Wiener filtering problem.

The optimal tracking problem as formulated here was first studied by Newton et al. and the results were presented in [2]. They assumed the plant to be single-input/single-output, linear, time-invariant, asymptotically stable, and characterized by a rational transfer function. The requirements for the random signal, the noise, and the optimization criterian were the same as those of the Wiener problem. They sought the design of a closed-loop cascade compensator of Figure 1.2. Since, as pointed out in [23], a direct formulation of the closed-loop problem was difficult, they reformulated the problem as an open-loop problem. Once the transfer function of the open-loop cascade compensator was obtained, the conversion to the closed-loop design was direct via transfer function manipulations.

The results of [2] were extended by Weston and Bongiorno [3] to the case of multi-input/multi-output plants and vector stochastic processes. More recently, Youla, Jabr and Bongiorno in [4,5] treated the closed-loop problem for both single-input/single-output and multi-input/multi-output systems. In addition to designing an optimal closed-loop compensator, they arrived at the necessary and sufficient conditions for its existence and the stability of the closed-loop system.

It should be noted that the work cited above was carried out strictly for stationary systems. That is, the

plants were linear and time-invariant, and the signals were stationary stochastic processes over semi-infinite intervals $(-\infty, t]$, characterized by rational power density spectra. The compensator design was therefore carried out in the frequency domain employing Wiener's frequency domain spectral factorization techniques. In the work presented in this dissertation, the above assumptions are totally relaxed. Thus the system admits time-varying plants, nonstationary stochastic processes, and finite observation intervals.

1.3 CONNECTION WITH THE LQG TRACKING PROBLEM

A particular stochastic optimal tracking problem called the "LQG Tracking Problem" has been studied extensively as a special case of the so called LQG (Linear Quadratic Gaussion) Regulator Problem [10]. In the LQG Tracking Problem, the plant and the stochastic processes are allowed to be nonstationary. The optimization criterian is the so called "integral of the mean-square error", together with a saturation like constraint on the "integral of the mean-square control".

As shown in [10], by augmenting the plant dynamics with the dynamics of the system generating the given stochastic process (called the reference system), the LQG Tracking Problem can be formulated in the context of the LQG Regulator Problem — the solution of which is well known [10]. The compensator structure of the LQG Tracking

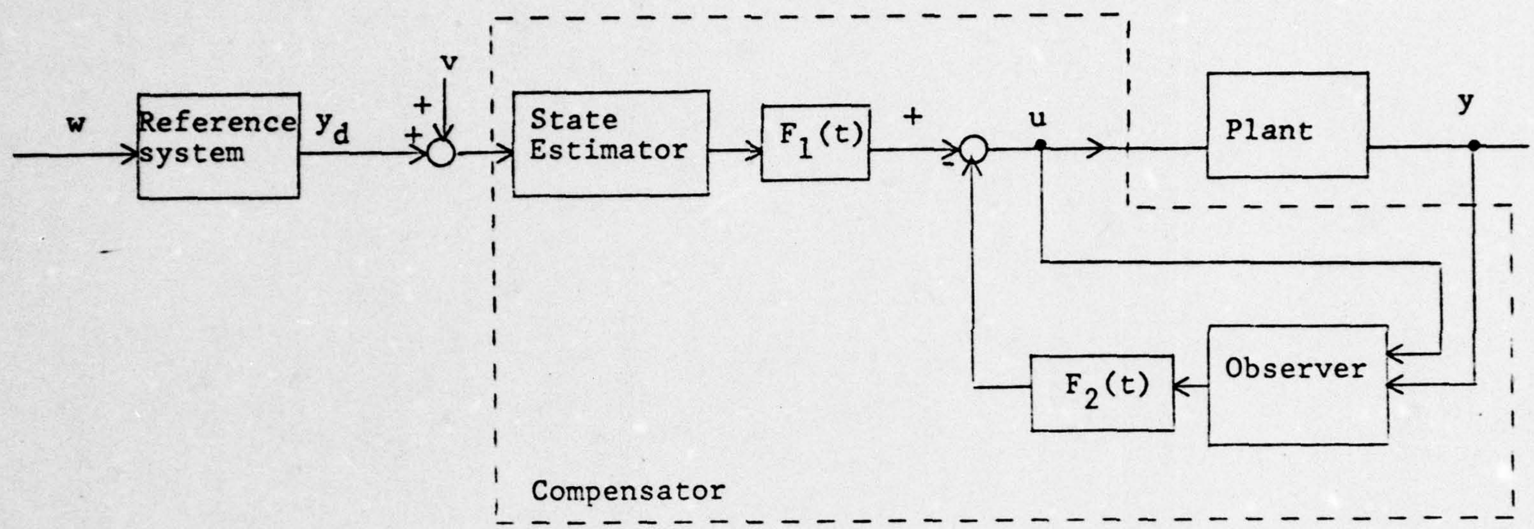
Problem is particularly interesting, and is shown in Figure 1.4 below.

In Figure 1.4 the state estimator estimates the state of the reference system generating the given stochastic process, and the observer reconstructs the state of the plant. The control is simply a linear combination of the states of the reference system and the plant. Note that the above system configuration, together with the feedback loop around the plant, is inherent in the LQG Tracking Problem, whereas in the Mean-square Tracking Problem, which is the subject of study here, the system configuration is totally free.

1.4 SUMMARY OF RESULTS AND CONTRIBUTIONS OF THIS RESEARCH

Both, the open-loop and the closed-loop, problems are treated in this dissertation. Due to the nonstationary nature of the system, it proves natural that the problems be formulated and solved in the time domain. Since extensive computer algorithms are available for state-space related computations, the final design of the optimal compensator is given in terms of its state-space realization.

First, the open-loop problem is formulated in the time domain. Variational techniques are then used to arrive at a Wiener-Hopf type integral equation which the optimal compensator must satisfy. Wiener's spectral factorization theory is employed to solve the resulting



THE LQG TRACKING PROBLEM

Figure 1.4

integral equation. New, special techniques to carry out the required factorization in the state space are developed in this disseration. These techniques enable one to arrive quite naturally at the space realization of the optimal compensator.

For the closed-loop case, as pointed out in [23], the problem formulation in the time domain is not so straightforward. To accomplish this end, Volterra's "Compositional Algebra", which was developed originally as an aid in the study of integral equations, is used. The compositional algebra is applied here to explicitly formulate the closed-loop problem. Once again the standard variational techniques are used to arrive at the necessary and sufficient conditions which the optimal closed-loop compensator must satisfy. The resulting integral equation is solved using the same techniques as for the open-loop problem to arrive at the state-space realization of the closed-loop compensator.

Contributions of this Research

This research addresses a fundamental unsolved problem in systems theory, which is, the generalization of the existing theory of stochastic optimal tracking to include nonstationary systems. Thus a unified theory of stochastic optimal tracking involving linear, time-varying plants and/or nonstationary processes is developed. Many of the previous results therefore become special

cases of the results developed here.

Another significant contribution of this research is the development of general state-space spectral factorization techniques. The implicit close relationship between the Riccati equation and the spectral factorization has been known since Kalman's work [12]. However use of equivalence transformations in state-space to accomplish spectral factorization clearly and explicitly demonstrates this relationship.

It is also demonstrated in this dissertation that the optimal minimum mean-space compensator can be separated into a Kalman State Estimator and a dynamic system. The previous work on this problem, and the techniques used fail to demonstrate this important property.

Finally, Volterra's compositional algebra techniques are applied here for the first time to time-varying feedback system optimizations. These techniques may prove to be useful to other problems of this nature.

1.5 ORGANIZATION OF THIS DISSERTATION

The next chapter, Chapter 2, deals with general system theory concepts, mathematical preliminaries and linear filtering theory. The topics presented are the ones which are used or referred to in the subsequent chapters. In Chapter 3, the open-loop Problem is formulated and solved. The optimal compensator is first solved for in terms of its impulse response, and then in terms of an

explicit state-space realization. In Chapter 4, the closed-loop problem is addressed. Again, both the impulse response and the state-space solutions of the optimal compensator are given. Chapter 5 addresses the problem of the open-loop and closed-loop stability of the system, and conditions are derived which guarantee the asymptotic stability of the system. Finally, in Chapter 6, a Conclusion is presented and recommendations are made for additional work in this area.

CHAPTER 2

SYSTEM THEORY AND MATHEMATICAL CONCEPTS

2.1 GENERAL

A complete understanding of the material in this dissertation requires a knowledge of general system theory, probability theory, stochastic processes, and some real analysis. Such knowledge will generally be assumed. In this chapter, however, basic definitions, concepts, and notation fundamental to other chapters will be presented. Additional concepts will be introduced as required in the sequel. The primary references for this chapter are D'Angelo [8], Popoulus [11], and Meditch [7].

2.2 LINEAR SYSTEMS

The state-space description of a linear p-input/q-output time-varying dynamical system is given by

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ y(t) &= C(t)x(t) + D(t)u(t) \qquad (2.1) \\ x(t_0) &= x_0 \end{aligned}$$

where $x(t)$ is an n-vector of state variables, $u(t)$ is a p-vector of input variables, the control; and $y(t)$ is a q-vector of output variables. Furthermore, $A(\cdot)$, $B(\cdot)$, $C(\cdot)$ and $D(\cdot)$ are matrix functions of the time variable t, and are of the order nxn, nxp, qxn and qxp, respectively. The

dot denotes the time derivative, t_0 indicates the initial time and x_0 the system initial state.

The above set of equations, for the sake of brevity, can also be written as $[A,B,C,D;\ t_0,x_0]$. If A(t) is continuous, and B(t) and u(t) are piecewise continuous for all time t, the solution of the above equation is

$$x(t) = \Phi(t,t_0)x(t_0) + \int_{t_0}^{t} \Phi(t,\tau)B(\tau)u(\tau)d\tau$$

for all $t \geq t_0$, where $\Phi(t,\tau)$ is the state transition matrix of the system. If the system is initially in the zero state, that is $x(t_0)=0$, the response of the output variable y(t) for $t \geq t_0$ is given by

$$y(t) = \int_{t_0}^{t} K(t,\tau)u(\tau)d\tau + D(t)u(t)$$

where

$$K(t,\tau) = C(t)\Phi(t,\tau)B(\tau)$$

The matrix $K(t,\tau)$ is called the impulse response matrix of the system.

A linear, relaxed, dynamical system is said to be causal or physically realizable if

$$K(t,\tau) = 0 \text{ for all } t<\tau$$

Equivalence

Two systems are said to be zero state equivalent if both systems have identical output when excited from the zero state with identical inputs. Two systems are zero-input equivalent if initial states (not necessarily equal) exist so that both systems have identical outputs with zero inputs.

There is a class of transformations that can be applied to the linear system characterized by Eq. (2-1), that always results in systems that are zero-state and zero-input equivalent. In particular, the equivalence transformation is defined by

$$\underline{x}(t) = T(t)x(t)$$

where T(t) is an nxn matrix, nonsingular and continuously differentiable on $[t_0, t]$. Applying the equivalence transformation to the system of Eq. (2.1) results in

$$\begin{aligned} \dot{\underline{x}}(t) &= \underline{A}(t)\underline{x}(t) + \underline{B}(t)u(t) \\ y(t) &= \underline{C}(t)\underline{x}(t) + \underline{D}(t)u(t) \end{aligned} \qquad (2.2)$$

where

$$\underline{A}(t) = [T(t)A(t) + \dot{T}(t)]T^{-1}(t)$$

$$\underline{B}(t) = T(t)B(t)$$

$$\underline{C}(t) = C(t)T^{-1}(t)$$

$$\underline{D}(t) = D(t)$$

The impulse response matrix of the tranformed system (2.2) is identical to that of the original system, whereas the state transition matrix of the transformed system is given by

$$\Phi(t,\tau) = T(t)\Phi(t,\tau)T^{-1}(\tau)$$

Adjoint Systems

A linear system S_a, characterized by its state transition matrix $\Phi_a(t,\tau)$ and its impulse response matrix $K_a(t,\tau)$ is the adjoint of a linear system S characterized by its state transition matrix $\Phi(t,\tau)$ and its impulse response matrix $K(t,\tau)$ if and only if

$$\text{(I)}\quad \Phi_a(t,\tau) = [\Phi^{-1}(t,\tau)]' = \Phi'(\tau,t)$$

and

$$\text{(II)}\quad K_a(t,\tau) = -K'(\tau,t)$$

for $t_0 \le \tau \le t$; where prime denotes the transpose. If a system satisfies (II) only, it is termed an input-output adjoint system.

Theorem 2.1

If a linear system S is characterized by

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$
$$y(t) = C(t)x(t) + D(t)u(t)$$

then the adjoint S_a of S satisfying properties (I) and (II) above is given by

$$\dot{x}_a(t) = -A'(t)x_a(t) \pm C'(t)u_a(t)$$
$$y_a(t) = \mp B'(t)x_a(t) + D'(t)u_a(t)$$

Proof

Refer to [8].

Inverse Systems

A system S^{-1} is an inverse system of a system $\underline{S}$ if, when $\underline{S}$ and $\underline{S}^{-1}$ are cascaded, the output of the combined system is identical with its input. Inverse systems are classified as pre-inverse systems and post-inverse systems. A post-inverse system for a p-input/q-output system can exist only if $p>q$, while a pre-inverse system of $\underline{S}$ can exist only if $p<q$.

Theorem 2.2

Consider a single input/single-output system $\underline{S}$ defined by

$$\dot{x}(t) = A(t)x(t) + b(t)u(t)$$
$$y(t) = c(t)x(t) + d(t)u(t)$$

where x, the state is an n-vector, and u and y are the scalar input and output respectively. If $d(t)\neq 0$ for all $t\epsilon[t_0,\infty)$, then the following set of equations represent an inverse $\underline{S}^{-1}$ of $\underline{S}$ in the sense that if y is the response of S to the input u on $[t_0,\infty)$ and the initial state x_0, then u is the response of $\underline{S}^{-1}$ to an input y on $[t_0,\infty)$ and the same initial state. Thus $\underline{S}^{-1}$ is given by

$$\dot{z}(t) = [A(t) - d^{-1}(t)b(t)c(t)]z(t) + d^{-1}(t)b(t)y(t)$$
$$u(t) = -d^{-1}(t)c(t)z(t) + d^{-1}(t)y(t)$$
$$z(t_0) = x_0$$

Proof

See Silverman [21].

2.3 STOCHASTIC PROCESSES AND LINEAR FILTERING

Consider a continuous-time measurement process model of the form

$$z(t) = y_d(t) + v(t),\ t \geq t_0 \tag{2.3}$$

Here $y_d(\cdot)$ is a Gaussian distributed stochastic process signal vector and $v(\cdot)$ is a zero mean Gaussian vector noise process with convariance

$$E[v(t)v'(\tau)] = R(t)\delta(t-\tau);\ t,\tau \geq t_0$$

The notation E denotes ensemble average or expectation, the prime denotes matrix transposition and $\delta(\cdot)$ is the Dirac-delta function. The quantity $z(\cdot)$ is called the measurement or the observation process. It is common to assume that the signal $y_d(\cdot)$ is generated by a finite-dimensional linear model

$$\begin{aligned} \dot{\psi}(t) &= A(t)\psi(t) + B(t)w(t) \\ y_d(t) &= C(t)\psi(t) \end{aligned} \tag{2.4}$$

for $t>t_0$. Here $A(\cdot)$, $B(\cdot)$, and $C(\cdot)$ are continuous matrix functions in the time variable t, and the process $w(\cdot)$ is zero-mean and Gaussian distributed with covariance

$$E[w(t)w'(\tau)] = Q(t)\delta(t-\tau);\ t,\tau>t_0$$

Furthermore, the initial state $\psi(t_0)$ of the system given by Eq (2.4) is a Gaussian random variable with mean ψ_0 and covariance P_0, viz.,

$$E[\psi(t_0)] = \psi_0$$
$$E[\psi(t_0)\psi'(t_0)] = P_0$$

Generally, we shall assume that $v(\cdot)$ and $w(\cdot)$ are uncorrelated; that is

$$E[v(t)w'(\tau)] = 0 \qquad (2.5)$$

for all t and τ.

When we consider the case where there is correlation between $x(\cdot)$ and $v(\cdot)$, we shall assume that

$$E[w(t)v'(\tau)] = \underline{R}(t)\delta(t-\tau) \qquad (2.6)$$

It should be noted that the description of $w(\cdot)$ and $v(\cdot)$ as white noise processes and the model description of the form (2.3) and (2.4) is by no means rigorous. A more rigorous representation is the stochastic differential equation [15]. However, the above notation is more familiar in linear systems theory and the results obtained

here under this representation do not differ from the corresponding results if stochastic differential equations are used.

We further assume that $Q(\cdot)$ is a nonnegative semi-definite matrix, and that $R(\cdot)$ is positive definite so that R^{-1} exists.

Let $S(t,r)$ be the impulse response of a causal, minimum-mean-square-error filter which operates on $z(\cdot)$ for $t \geq t_0$ with $\hat{y}_d(\cdot)$ as its output. Then

$$\hat{y}_d(t) = \int_{t_0}^{t} S(t,r)z(r)dr$$

is a linear functional of the measurement process $z(\tau)$, $t_0 \leq \tau \leq t$, that minimizes

$$E\Big[[y_d(t)-\hat{y}_d(t)]' \, [y_d(t)-\hat{y}_d(t)]\Big]$$

It is well-known that $S(t,r)$ satisfies the Wiener-Hopf type equation

$$\int_{-\alpha}^{+\alpha} S(t,r)E[z(r)z(a)]\,dx = E[y_d(t)z(a)] \tag{2.7}$$

for all $t_0 \leq a \leq t$

Theorem 2.3

Continuous-time Kalman-Bucy Filter. Given the model (2.3), the filter S(t,n) defined by Eq. (2.7) is represented by the matrix differential equation

$$\dot{\hat{\psi}}(t) = [A(t) - K(t)C(t)]\hat{\psi}(t) + K(t)z(t) \qquad (2.8)$$

$$\hat{y}_d(t) = C(t)\psi(t)$$

where

$$K(t) = P(t)C(t)R^{-1}(t) \qquad (2.9a)$$

in the case where there is no correlation between input and output noise processes, i.e., (2.5) applies, and

$$K(t) = [P(t)C(t) + B(t)\underline{R}(t)]R^{-1}(t) \qquad (2.9b)$$

in the situation where (2.6) applies. The matrix P(t) defined as $E\left[[y_d-\hat{y}_d][y_d-\hat{y}_d]'\right]$ is the positive definite solution of the matrix Riccati differential equation

$$\dot{P}(t) = A(t)P(t) + P(t)A'(t) - K(t)R(t)K'(t) + B(t)Q(t)B'(t)$$

subject to the initial condition

$$P(t_0) = [E\ \psi(t_0)\psi'(t_0)] = P_0$$

Proof

Refer to [7,12].

Theorem 2.4

(The Innovations Theorem.) Given (2.3), the innovations process $\upsilon(\cdot)$ defined by

$$\upsilon(t) = z(t) - \hat{y}_d(t),\ t \geq t_0$$

is a zero mean, white Gaussian noise process with the same covariance as $v(\cdot)$, i.e.,

$$E[\upsilon(t)\upsilon'(\tau)] = R(t)\delta(t-\tau),$$

Furthermore, the processes $z(\cdot)$ and $\upsilon(\cdot)$ are related by a causal linear operation.

Refer to [13,14] for a proof and detailed discussion of Theorem 2.4.

<u>Comment</u>

From Theorem 2.4 $\upsilon(\cdot)$ and $z(\cdot)$ are related by a causal linear operation. That is, $\upsilon(\cdot)$ can be obtained from $z(\cdot)$ by passing $z(\cdot)$ through a linear causal filter, commonly called the "Whitening filter". Thus

$$\upsilon(\cdot) = \int_{-\infty}^{+\infty} W_w(t,r)z(r)dx \tag{2.10}$$

where $W_w(t,r)$ is the impulse response of the whitening filter.

Conversely, $z(\cdot)$ can be obtained from $\upsilon(\cdot)$ by passing $\upsilon(\cdot)$ through the inverse* of the filter $W_w(t,r)$. Denoting the inverse of $W_w(t,r)$ by $W_w^{-1}(t,r)$ in the sense that

---

*The inverse here is defined in the sense of Section 2.2. The existence of the inverse of $W_w(t,x)$ is discussed in Kailath [13].

$$\int_{-\infty}^{+\infty} W_w(t,\eta)W_w^{-1}(\eta,r)d\eta = \int W_w^{-1}(t,\eta)W_w(\eta,r)d\eta = I_n$$

where I_n is an identity matrix of appropriate dimensions, we have

$$z(t) = \int_{-\infty}^{+\infty} W_w^{-1}(t,r)\ \upsilon(r)dx \tag{2.11}$$

CHAPTER 3

THE OPEN-LOOP PROBLEM

3.1 GENERAL

In this chapter the open-loop optimal tracking problem is formulated and solved. Since the systems under study are nonstationary, it seems natural that we use time-domain techniques for the analysis and design. The final design of the open-loop cascade compensator is presented in terms of its state-space representations. The main results of this chapter are presented in the form of Theorems 3.1 and 3.5. The organization of the chapter is as follows.

Section 3.2 is devoted to the state-space formulation of the problem. In Section 3.3, a necessary and sufficient condition which the optimal open-loop cascade compensator must satisfy is derived. This condition turns out to be a type of nonstationary Wiener-Hopf integral equation. The integral equation of Section 3.3 is then solved in Section 3.4 to arrive at both the impulse response and the state-space representation of the open-loop cascade compensator. In Section 3.4, we also develop the state-space spectral factorization techniques and illustrate them via examples. In Section 3.5, a brief discussion and critique of the results is presented.

3.2 PROBLEM STATEMENT

Consider a linear, time-varying, single-input/single-output plant DS1 (Figure 3.1) defined by

$$\text{DS1:}\quad \begin{aligned} \dot{x}(t) &= F(t)x(t) + g(t)u(t) && (3.1a)\\ y(t) &= h(t)x(t) && (3.1b)\\ x(t_0) &= 0 && (3.1c) \end{aligned}$$

where $x(t)$ is an n-vector, the state; u is the control input and y is the system output, both scalars, and $F(\cdot)$, $g(\cdot)$ and $h(\cdot)$ are real, continuous matrix functions of time t which are nxn, nx1, and 1xn, respectively. The dot here denotes the time derivative and t_0 is the initial time.

Let $y_d(t)$ be the reference signal which the output $y(t)$ of the plant DS1 is required to track optimally. We assume that the signal $y_d(\cdot)$ is generated by a finite-dimensional, linear model of the form

$$\text{DS2:}\quad \begin{aligned} \dot{\psi}(t) &= A(t)\psi(t) + b(t)w(t)\\ y_d(t) &= c(t)\psi(t) \end{aligned} \qquad (3.2a)$$

for $t \geq t_0$. Here, ψ is an m-vector, the state; $w(\cdot)$ and $y_d(\cdot)$ are scalars, and $A(\cdot)$, $b(\cdot)$, and $c(\cdot)$ are continuous matrix functions of t, which are mxm, mx1, and 1xm, respectively. Furthermore, the process $\{w(t),\ t \geq t_0\}$ is a zero mean, Gaussian, white noise process with covariance

$$E[w(t)w(\tau)] = Q(t)\delta(t-\tau) \qquad (3.3)$$

and the initial state $\psi(t_0)$ of the system given by Eq. (3.2) is a Gaussian distributed random variable with mean ψ_0 and covariance P_0,

$$E[\psi(t_0)] = \psi_0 \qquad (3.4a)$$

$$E[\psi(t_0)\psi'(t_0)] = P_0 \qquad (3.4b)$$

The system defined by Eq. (3.2) will be termed the reference system DS2.

The signal available for input to the plant DS1 is modeled as

$$z(t) = y_d(t) + v(t) \qquad (3.5)$$

where $\{v(t),\ t \geq t_0\}$ is a zero mean, Gaussian, white noise process with variance

$$E[v(t)\ v(\tau)] = R(t)\delta(t-\tau),\ t \geq t_0 \qquad (3.6)$$

We assume that Q(t) and R(t) are both continuous, that Q(t) is nonnegative, and that R(t) is positive definite such that $R^{-1}(t)$ exists.

The tracking error $y_e(t)$ is defined by

$$y_e(t) = y(t) - y_d(t) \qquad (3.7)$$

As a measure of the system performance, we choose a suitably weighted sum of the mean-square tracking error

y_e and the mean-square control effort u. Denoting the index of system performance (the cost functional) by L, we thus have

$$L = E[y_e^2(t)] + k(t)E[u^2(t)] \tag{3.8}$$

where $k(t)>0$ is a Lagrange multiplier. A suitable value of k is chosen in the last stages of the design process to satisfy a saturation-like constraint on u, of the form

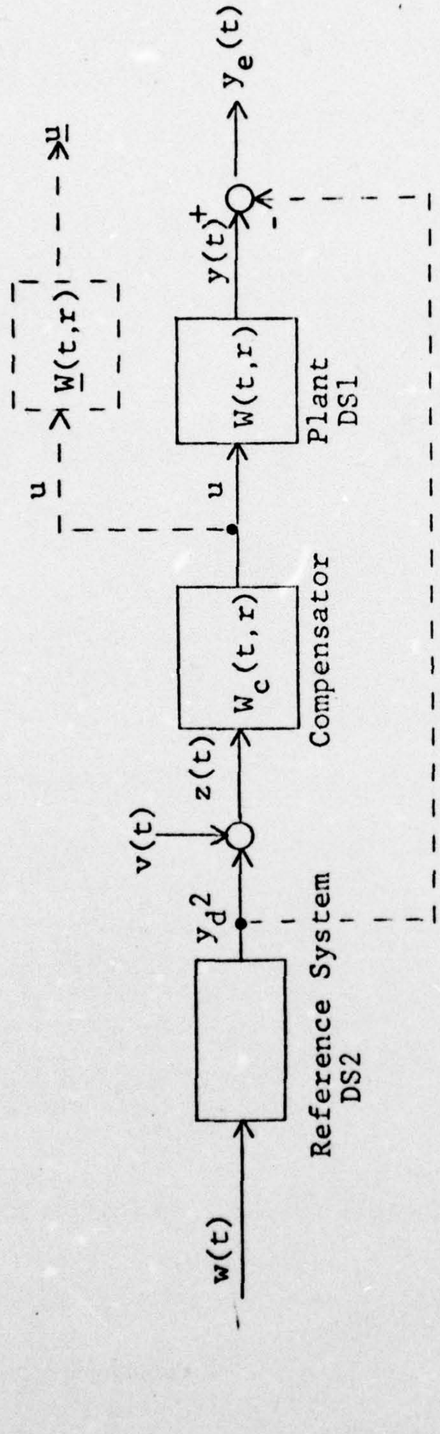
$$E[u^2(t)] \le \beta$$

where $\beta>0$. Our problem can now be stated as follows: Given the plant DS1, the reference system DS2, the relevent noise statistics and the constraints on u, design and realize in state-space an optimal open-loop cascade compensator such that the cost functional L given by Eq. (3.8) is minimized.

3.3 DERIVATION OF THE WIENER-HOPF INTEGRAL EQUATION

A block diagram of the system with which we are concerned in this chapter is shown below in Figure 3.1. In the diagram, $W(t,r)$ and $W_c(t,r)$ are the impulse response functions of the plant and the optimal compensator, respectively.

To keep track of variables, as will become apparent subsequently, we shall use $\underline{u}$ instead of u in Eq. (3.8), where $\underline{u}$ is a linear functional of u. That is



OPEN-LOOP TRACKING PROBLEM

Figure 3.1

$$\underline{u} = \int_{-\infty}^{+\infty} \underline{W}(t,r)u(r)dr, \quad t \geq t_0.$$

Later, we can always choose $\underline{W}(t,x)$ to be the identity system ($\underline{W}(t,r) = \delta(t-r)$) such that $\underline{u}=u$. Hence, in view of Figure 3.1, a modified form of Eq. (3.8) is

$$L_1 = E[y_d^2(t) - 2y_d(t)y(t) + y^2(t)] + k(t)E[\underline{u}^2(t)] \quad (3.9)$$

It is clear that $\underline{u}(t)$ and $y(t)$ can be written as

$$\underline{u}(t) = \int_{-\infty}^{+\infty} dp \int_{-\infty}^{+\infty} ds \; \underline{W}(t,p)W_c(p,s)z(s) \quad (3.10)$$

and

$$y(t) = \int_{-\infty}^{+\infty} dp \int_{-\infty}^{+\infty} ds \; W(t,p)W_c(p,s)z(s) \quad (3.11)$$

respectively, where from causality, $t_0 \leq s \leq p$, $t_0 \leq p \leq t$, and $t \geq t_0$. Substituting from Eqs. (3.10) and (3.11) into Eq. (3.9), we get

$$L_1 = E\left[y_d^2(t) - 2y_d(t)\int_{-\infty}^{+\infty} dq \int_{-\infty}^{+\infty} dr \; W(t,q)W_c(q,r)z(r)\right.$$

$$+ \left\{ \int_{-\infty}^{+\infty} dp \int_{-\infty}^{+\infty} ds \; W(t,p)W_c(p,s)z(s) \right\} x$$

$$\left\{ \int_{-\infty}^{+\infty} dq \int_{-\infty}^{+\infty} dr \; W(t,q)W_c(q,r)z(r) \right\}$$

$$+ \quad k \left\{ \int_{-\infty}^{+\infty} dp \int_{-\infty}^{+\infty} ds \; \underline{W}(t,p)W_c(p,s)z(s) \right\} \; x$$

$$\left. \left\{ \int_{-\infty}^{+\infty} dq \int_{-\infty}^{+\infty} dr \; \underline{W}(t,q)W_c(q,r)z(r) \right\} \right] \qquad (3.12)$$

Equation (3.12) can be rewritten as

$$L_1 = E\left[\; y_d^2(t) \; - \; 2y_d(t) \int_{-\infty}^{+\infty} dq \int_{-\infty}^{+\infty} dr \; W(t,q)W_c(q,r)z(r) \right.$$

$$+ \int_{-\infty}^{+\infty} dp \int_{-\infty}^{+\infty} ds \int_{-\infty}^{+\infty} dq \int_{-\infty}^{+\infty} dr \; W(t,p)W_c(p,s)W(t,q)$$

$$W_c(q,r)z(s)z(r)$$

$$+ \; k \int_{-\infty}^{+\infty} dp \int_{-\infty}^{+\infty} ds \int_{-\infty}^{+\infty} dq \int_{-\infty}^{+\infty} dr \; \underline{W}(t,p)W_c(p,s)\underline{W}(t,q)$$

$$\left. W_c(q,r)z(s)z(r) \; \right] \qquad (3.13)$$

We now assume the existence of a physically realizable compensator $W_{co}(t,s)$ that minimizes Eq. (3.9). Next, we proceed to use a well known variational technique to arrive at $W_{co}(t,s)$. We thus let

$$W_c(t,s) = W_{co}(t,s) + eW_e(t,s) \tag{3.14}$$

where $W_e(t,s)$ is the variation of $W_c(t,s)$. Substituting Eq. (3.14) into Eq. (3.13), we have

$$L_1 = E\Bigg[y_d^2(t) - 2y_d(t)\int_{-\infty}^{+\infty} dq \int_{-\infty}^{+\infty} dr\ W(t,q)\Big\{ W_{co}(q,r) + eW_e(q,r)\Big\} z(r)$$

$$+ \int_{-\infty}^{+\infty} dp \int_{-\infty}^{+\infty} ds \int_{-\infty}^{+\infty} dq \int_{-\infty}^{+\infty} dr\ W(t,p)W(t,q)\Big\{ W_{co}(p,s)W_{co}(q,r)$$

$$+ eW_{co}(p,s)W_e(q,r)$$

$$+ eW_e(p,s)W_{co}(q,r) + e^2W(p,s)\Big\} W(q,r)z(s)z(r)$$

$$+ k\int_{-\infty}^{+\infty} dp \int_{-\infty}^{+\infty} ds \int_{-\infty}^{+\infty} dq \int_{-\infty}^{+\infty} dr\ \underline{W}(t,p)\underline{W}(t,q)\Big\{ W_{co}(p,s)W_{co}(q,r)$$

$$+ eW_{co}(p,s)W_e(q,r)$$

$$+ eW_e(p,s)W_{co}(q,r) + e^2W(p,s)\Big\} W(q,r)z(s)z(r)\Bigg] \tag{3.15}$$

A necessary condition for L_1 to be a minimum is

$$\left.\frac{\partial L_1}{\partial e}\right|_{e=0} = 0$$

Carrying out the required differentiation and recognizing that p is interchangeable with q, and that s

is interchangeable with r, we obtain

$$\left.\frac{\partial L_1}{\partial e}\right|_{e=0} = E\Bigg[-2y_d(t)\int_{-\infty}^{+\infty} dq \int_{-\infty}^{+\infty} dr\; W(t,q)W(q,r)z(r)$$

$$+\, 2\int_{-\infty}^{+\infty} dp \int_{-\infty}^{+\infty} ds \int_{-\infty}^{+\infty} dq \int_{-\infty}^{+\infty} dr\; W(t,p)W(t,q)$$

$$W_{co}(p,s)W_e(q,r)z(s)z(r)$$

$$+\, 2k\int_{-\infty}^{+\infty} dp \int_{-\infty}^{+\infty} ds \int_{-\infty}^{+\infty} dq \int_{-\infty}^{+\infty} dr\; \underline{W}(t,p)\underline{W}(t,q)$$

$$W_{co}(p,s)W_e(q,r)z(s)z(r)\Bigg] = 0 \qquad (3.16)$$

If we interchange the order of the integral and expectation operators in Eq. (3.16) and use the fact that $W_e(q,r) = 0$ for $r<t_0$ and that $W_e(q,r)$ is arbitrary for $t_0 \le r \le q \le t_0$, we are led to the following result.

<u>Theorem 3.1</u>

For a physically realizable (causal) compensator W_{co} to minimize L_1, it is necessary and sufficient that it satisfy

$$\int_{-\infty}^{+\infty} dp \int_{-\infty}^{+\infty} ds \int_{-\infty}^{+\infty} dr\; W(t,r)W(t,p)W_{co}(p,s)E[z(s)z(r)]$$

$$+\, k\int_{-\infty}^{+\infty} dp \int_{-\infty}^{+\infty} ds \int_{-\infty}^{+\infty} dr\; \underline{W}(t,r)\underline{W}(t,p)W_{co}(p,s)E[z(s)z(r)]$$

$$= \int_{-\infty}^{+\infty} dr\ W(t,r)E[y_d(t)z(r)]$$

for all $t>t_0$, $t_0<s<p$, $t_0<p<t$, and $t_0<r<t$. (3.17)

Proof

Necessity is immediate from above. Sufficiency follows from consideration of the second variation.

3.4 SOLUTION OF THE INTEGRAL EQUATION

Solution of the Wiener-Hopf type integral equation of Section 3.3 for a physically realizable $W_{co}(t,r)$ is developed in a number of steps.

First, the Innovations Theorem, Theorem 2.4 and the whitening filter given by Eqs. (2.10) and (2.11) are used to convert the Wiener-Hopf type integral equation involving covariances into a Wiener-Hopf type integral equation involving operators only. This conversion process is illustrated in Lemmas 3.1, 3.2 and Theorem 3.2. Second, the resulting integral equation which now involves adjoint operators, is solved for a physically realizable operator representing the optimal compensator. Wiener's spectral factorization techniques are used during this step, and the optimal compensator arrived at is in terms of its impulse response $W_{co}(t,r)$. Third, an explicit state-space realization of $W_{co}(t,r)$ is obtained. This realization process is developed in Theorems 3.3 and 3.4.

Lemma 3.1

$$\{E[y_d(t)\upsilon(a)]R^{-1}(a)\} * W_w(t,a) = S(t,a) \quad (3.18)$$

where $\upsilon(\cdot)$, $W_w(\cdot,\cdot)$ and $S(\cdot,\cdot)$ are respectively, the innovations process, the whitening filter and the Kalman-Bucy filter associated with the process $z(\cdot)$, and the asterisk denotes the convolution of linear operators.

Proof

Let $\hat{y}_d(t)$ be the filtered estimate of $y_d(t)$. Thus

$$\hat{y}_d(t) = \int_{-\alpha}^{+\alpha} S(t,r)z(r)dr, \; t_0 \leq r \leq t \quad (3.19)$$

where $S(t,r)$ satisfies Eq. (2.7). Since $z(\cdot)$ and $\upsilon(\cdot)$ are informationally equivalent to each other, $\hat{y}_d(t)$ is also given by

$$\hat{y}_d(t) = \int_{-\alpha}^{+\alpha} \hat{S}(t,r)\upsilon(r)dr, \; t_0 \leq r \leq t \quad (3.20)$$

where $\hat{S}(t,r)$ satisfies the integral equation

$$\int_{-\alpha}^{+\alpha} \hat{S}(t,r)E[\upsilon(r)\upsilon(a)]dr = E[y_d(t)\upsilon(a)], \; t_0 \leq a \leq t \quad (3.21)$$

From the innovations theorem,

$$E[\upsilon(r)\upsilon(a)] = E[v(r)v(a)] = R(r)\delta(r-a) \quad (3.22)$$

Substituting Eq. (3.20) into Eq. (3.19) and carrying out the integration with respect to r, we have

$$\hat{S}(t,a)R(a) = E[y_d(t)\upsilon(a)]$$

or

$$\hat{S}(t,a) = E[y_d(t)\upsilon(a)]R^{-1}(a) \qquad (3.23)$$

Recall from Chapter 2, Eq. (2.11), that

$$z(t) = \int_{-\alpha}^{+\alpha} W_w^{-1}(t,r)\upsilon(r)dr, \quad t_0 \le r \le t \qquad (3.24)$$

If we substitute Eq. (3.24) into Eq. (3.19) and compare the result with Eq. (3.20), we note that

$$S(t,r) * W_w^{-1}(t,r) = \hat{S}(t,r)$$

or

$$S(t,r) = \hat{S}(t,r) * W_w(t,r) \qquad (3.25)$$

Thus, from Eqs. (3.23) and (3.25), we have

$$S(t,r) = \left\{E[y_d(t)\upsilon(r)]R^{-1}(r)\right\} * W_w(t,r)$$

Q.E.D.

<u>Lemma 3.2</u>

The integral equation (3.17) and the following integral equation are equivalent:

$$\int_{-\alpha}^{+\alpha} dr\ W(t,r)\left\{\int_{-\alpha}^{+\alpha} dp \int_{-\alpha}^{+\alpha} ds\ W(t,p)W_{co}(p,s)[W_w^{-1}(s,r)R(r)]\right\}$$

$$+\ k\int_{-\alpha}^{+\alpha} dr\ \underline{W}(t,r)\left\{\int_{-\alpha}^{+\alpha} dp \int_{-\alpha}^{+\alpha} ds\ \underline{W}(t,p)W_{co}(p,s)[W_w^{-1}(s,r)R(r)]\right\}$$

$$= \int_{-\alpha}^{+\alpha} dr\ W(t,r)E[y_d(t)\upsilon(r)] \qquad (3.26)$$

for $t \geq p \geq s \geq t_0$, $t_0 \leq r \leq t$.

Proof

Substituting Eq. (3.24) into Eq. (3.17), we obtain

$$\int_{-\alpha}^{+\alpha} dp \int_{-\alpha}^{+\alpha} ds \int_{-\alpha}^{+\alpha} dr\ W(t,r)W(t,p)W_{co}(p,s)\,x$$

$$E\left[\int_{-\alpha}^{+\alpha} d\xi \int_{-\alpha}^{+\alpha} d\eta W_w^{-1}(s,\xi)\upsilon(\xi)\upsilon(\eta)W_w^{-1}(r,\eta)\right]$$

$$+\ k\int_{-\alpha}^{+\alpha} dp \int_{-\alpha}^{+\alpha} ds \int_{-\alpha}^{+\alpha} dr\ \underline{W}(t,r)\underline{W}(t,p)W_{co}(p,s)\,x$$

$$E\left[\int_{-\alpha}^{+\alpha} d\xi \int_{-\alpha}^{+\alpha} d\eta W_w^{-1}(s,\xi)\upsilon(\xi)\upsilon(\eta)W_w^{-1}(r,\eta)\right]$$

$$= \int_{-\alpha}^{+\alpha} drW(t,r)E\left\{\int_{-\alpha}^{+\alpha} d\eta\ y_d(t)\upsilon(\eta)W_w^{-1}(r,\eta)\right\}$$

Next, we interchange the expectation and integral operators, substitute Eq. (3.22) into the above equation and carry out the integration of the resulting delta function with respect to ξ to obtain

$$\int_{-\alpha}^{+\alpha} dp \int_{-\alpha}^{+\alpha} ds \int_{-\alpha}^{+\alpha} dr \int_{-\alpha}^{+\alpha} d\eta W(t,r) W(t,p) W_{co}(p,s) W_w^{-1}(s,\eta) R(\eta) W_w^{-1}(r,\eta)$$

$$+ k \int_{-\alpha}^{+\alpha} dp \int_{-\alpha}^{+\alpha} ds \int_{-\alpha}^{+\alpha} dr \int_{-\alpha}^{+\alpha} d\eta \underline{W}(t,r) \underline{W}(t,p) W_{co}(p,s) W_w^{-1}(s,\eta) R(\eta) W_w^{-1}(r,\eta)$$

$$= \int_{-\alpha}^{+\alpha} dr \int_{-\alpha}^{+\alpha} d\eta W(t,r) E[y_d(t) \upsilon(\eta)] W_w^{-1}(r,\eta) \qquad (3.27)$$

Equation (3.27) can be rewritten as

$$\int_{-\alpha}^{+\alpha} d\eta W_w^{-1}(r,\eta) \Bigg\{ \int_{-\alpha}^{+\alpha} dr \int_{-\alpha}^{+\alpha} dp \int_{-\alpha}^{+\alpha} ds W(t,r) W(t,p) W_{co}(p,s) W_w^{-1}(s,\eta) R(\eta)$$

$$+ k \int_{-\alpha}^{+\alpha} dr \int_{-\alpha}^{+\alpha} dp \int_{-\alpha}^{+\alpha} ds \underline{W}(t,r) \underline{W}(t,p) W_{co}(p,s) W_w^{-1}(s,\eta) R(\eta)$$

$$- \int_{-\alpha}^{+\alpha} W(t,r) E[y_d(t) \upsilon(\eta)] \Bigg\} = 0$$

Note that $W_w^{-1}(r,\eta)$ in the above equation is arbitrary in the sense that the equation holds true for any $W_w^{-1}(r,\eta)$ over the interval $t_0 \le \eta \le \upsilon \le t$. Obviously then, the above equation reduces to

$$\int_{-\infty}^{+\infty} dr \int_{-\infty}^{+\infty} dp \int_{-\infty}^{+\infty} ds\ W(t,r)W(t,p)W_{co}(p,s)W_w^{-1}(s,r)R(r)$$

$$+ k\int_{-\infty}^{+\infty} dr \int_{-\infty}^{+\infty} dp \int_{-\infty}^{+\infty} ds\ \underline{W}(t,r)\underline{W}(t,p)W_{co}(p,s)W_w^{-1}(s,r)R(r)$$

$$= \int_{-\infty}^{+\infty} dr\ W(t,r)E[y_d(t)\upsilon(r)] \tag{3.28}$$

Equation (3.26) follows immediately from Eq.(3.28).

Q.E.D.

<u>Theorem 3.2</u>

The integral equation (3.28) and the following operator equation are equivalent:

$$W^a * W * W_{co} + k\underline{W}^a * \underline{W} * W_{co} = W^a * S \tag{3.29}$$

where S is the Kalman-Bucy filter associated with the process $z(\cdot)$, and is given by Eqs. (2.8) and (2.9).

<u>Proof</u>

The bracketed portion of the first term on the left-hand side (LHS) of Eq. (3.26) is a convolution of the linear operators $W(t,a)$, $W_{co}(t,a)$ and $\{W_w^{-1}(t,a)R(a)\}$. We denote the resulting operator by $W_1(t,a)$, i.e.,

$$W_1(t,a) = W(t,a) * W_{co}(t,a) * \{W_w^{-1}(t,a)R(a)\}$$

and write the above term in Eq. (3.29) as

$$\int_{-\infty}^{+\infty} da \; W(t,a)W_1(t,a)$$

Now let $W^a(t,a)$ be the adjoint of the system $W(t,a)$. Since $W(t,a) = W^a(a,t)$, the above expression can be written as the convolution of two linear operators — $W^a(a,t)$ which is physically unrealizable (noncausal), and $W_1(t,a)$ which is physically realizable (causal). Hence, the first term of the LHS of Eq. (3.29) can be written as

$$W^a * W * W_{co} * W_w^{-1}R$$

Using the same argument for the rest of the terms of Eq. (3.29), this equation becomes

$$W^a * W * W_{co} * W_w^{-1}R + k\underline{W}^a * \underline{W} * W_{co} * W_w^{-1}R = W^a(a,t) * \{E[y_d(t)\upsilon(a)]\}$$

or

$$W^a * W * W_{co} * W_w^{-1} + k\underline{W}^a * \underline{W} * W_{co} * W_w^{-1} = W^a * E[y_d(t)\upsilon(a)]R^{-1}$$

or

$$W^a * W * W_{co} + k\underline{W}^a * \underline{W} * W_{co} = W^a(a,t) * \{E[y_d(t)\upsilon(a)]R^{-1}(a)\} * W_w(t,a)$$

If we substitute the results of Lemma 3.1 into the above equation, we obtain immediately the result

$$W^a * W * W_{co} + k\underline{W}^a * \underline{W} * W_{co} = W^a * S \qquad (3.29)$$

for $t \geq t_0$.

Q.E.D.

Equation (3.29) holds only for $t \geq t_0$ and involves linear operators and their adjoints. It can therefore be solved using Wiener's spectral factorization techniques [1,2]. The solution is developed in Eqs. (3.31) through (3.35) below. Before we proceed, we substitute for $\underline{W}$ an identity system with $\delta(t-\tau)$ as its impulse response. This step is in line with the original formulation of the problem. Equation (3.29) thus reduces to

$$(W^a * W + k) * W_{co} = W^a * S \tag{3.30}$$

Let

$$W^a * W + k = M^a * M \tag{3.31}$$

where M^a is the physically nonrealizable (noncausal) adjoint of some operator M. Thus

$$M^a * M * W_{co} = W^a * S \tag{3.32}$$

or

$$M * W_{co} = [M^a]^{-1} * W^a * S \tag{3.33}$$

where $[M^a]^{-1}$ is the inverse of the system M^a in the sense of Theorem 2.2.

Next, the right hand side of Eq. (3.33) is written as a "sum" of two systems described by the operators $\underline{L}^a$ and L_2, where L_2 is causal and $\underline{L}^a$ is noncausal. That is

$$[M^a]^{-1} * W^a * S = \underline{L}^a + L_2 \tag{3.34}$$

Thus

$$M * W_{co} = L_2$$

or

$$W_{co} = M^{-1} * L_2 \tag{3.35}$$

Equation (3.35) gives the impulse response of the open-loop cascade compensator W_{co} in terms of the parameters of the systems DS1, DS2, the noise processes w, v, and the Lagrange multiplier k(t), all imbedded into M^{-1} and L_2. The next step is to lay bare the structure of M^{-1} and L_2 in terms of the system parameters, and obtain a realization of W_{co} in the more useful state-space form. To accomplish this, we carry out in the sequel, the mathematical operations described by Eqs. (3.31) through (3.35) directly in state-space. To curb the proliferation of symbols, we denote by u_1 and y_1 the scalar input and output, respectively, of any given dynamic system. Furthermore, we use the symbol "$\longleftrightarrow$" to denote the equivalence between the state-space and the impulse response representations of systems.

<u>Theorem 3.3</u>

Let $(F,g,h;t_0,0)$ be a realization of W, i.e.,

$$W \longleftrightarrow (F,g,h;t_0,0)$$

Furthermore, let

$$W^a * W + k = M^a * M$$

where W^a is the noncausal adjoint of W, M^a is the noncausal adjoint of some operator M, and $k>0$. Then $(F,g,\underline{h},\underline{\ell};t_0,0)$ is a realization of M, where

$$\underline{\ell} = \sqrt{k}$$

and

$$\underline{h} = \frac{1}{\sqrt{k}} g'T$$

Here T is a symmetric, nonnegative definite nxn matrix satisfying the Riccati Equation

$$\dot{T} = -\frac{1}{k} Tgg'T + TF + F'T + h'h \qquad (3.36)$$

subject to the initial condition

$$T(t_0) = 0_{nxn}$$

Proof

From Theorem 2.2.

$$W^a \longleftrightarrow (-F',h',-g',t_0,0)$$

Denoting the states of W and W^a by x and x_a respectively, we have

$$
W^a * W + k \longleftrightarrow \begin{cases} \begin{bmatrix} \dot{x} \\ \hline \dot{x}_a \end{bmatrix} = \left[\begin{array}{c|c} F & 0 \\ \hline h'h & -F' \end{array}\right] \begin{bmatrix} x \\ \hline x_a \end{bmatrix} + \begin{bmatrix} g \\ \hline 0 \end{bmatrix} u_1 \\ y_1 = \left[\begin{array}{c|c} 0 & -g' \end{array}\right] \begin{bmatrix} x \\ \hline x_a \end{bmatrix} + ku_1 \\ \begin{bmatrix} x(t_0) \\ \hline x_a(t_0) \end{bmatrix} = \begin{bmatrix} 0 \\ \hline 0 \end{bmatrix} \end{cases} \quad (3.37)
$$

We let

$$
M \longleftrightarrow (\underline{f}, \underline{g}, \underline{h}, \underline{\ell}; t_0, 0)
$$

Therefore

$$
M^a \longleftrightarrow (-\underline{F}', \underline{h}', -\underline{g}', \underline{\ell}; t_0, 0)
$$

If we denote by $\underline{s}$ *and* $\underline{s}_a$ *the state vectors of* M and M^a, respectively, we can write

$$
M^a * M \longleftrightarrow \begin{Bmatrix} \begin{bmatrix} \dot{\underline{s}} \\ \hline \dot{\underline{s}}_a \end{bmatrix} \quad \left[\begin{array}{c|c} \underline{F} & 0 \\ \hline \underline{h}'\underline{h} & -\underline{F}' \end{array}\right] \begin{bmatrix} \underline{s} \\ \hline \underline{s}_a \end{bmatrix} + \begin{bmatrix} \underline{g} \\ \hline h'\underline{\ell} \end{bmatrix} u_1 \\ y_1 = \left[\begin{array}{c|c} \underline{\ell h} & -\underline{g}' \end{array}\right] \begin{bmatrix} \underline{s} \\ \underline{s}_a \end{bmatrix} + \underline{\ell}^2 u_1 \\ \begin{bmatrix} \underline{s}(t_0) \\ \hline \underline{s}_a(t_0) \end{bmatrix} = \begin{bmatrix} 0 \\ \hline 0 \end{bmatrix} \end{Bmatrix} \quad (3.38)
$$

Equation (3.38) obviously has a slightly different form than that of Eq. (3.37). To find $\underline{F}, \underline{g}, \underline{h}$, and $\underline{\ell}$, we transform Eq. (3.37) into the same form as Eq. (3.38). The following equivalence transformation is used for this purpose,

$$\hat{T}(t) = \left[\begin{array}{c|c} I_{nxn} & 0_{nxn} \\ \hline T(t)_{nxn} & I_{nxn} \end{array}\right]$$

so that

$$\hat{T}^{-1}(t) = \left[\begin{array}{c|c} I & 0 \\ \hline -T(t) & I \end{array}\right], \text{ and } \frac{d}{dt}\hat{T}(t) = \left[\begin{array}{c|c} 0 & 0 \\ \hline \dot{T}(t) & 0 \end{array}\right]$$

From Section 2.2, we thus have

$$(W^a * W + k) \longleftrightarrow \left\{ \begin{array}{l} \left[\begin{array}{c} \dot{\underline{x}} \\ \hline \dot{\underline{x}}_a \end{array}\right] = \left[\begin{array}{c|c} F & 0 \\ \hline \begin{matrix} -\dot{T} + TF \\ h'h + F'T \end{matrix} & -F^{1} \end{array}\right] \left[\begin{array}{c} \underline{x} \\ \hline \underline{x}_a \end{array}\right] + \left[\begin{array}{c} g \\ \hline Tg \end{array}\right] u_1 \\ \\ y_1 = \left[\begin{array}{c|c} g'T & -g' \end{array}\right] \left[\begin{array}{c} \underline{x} \\ \hline \underline{x}_a \end{array}\right] + ku_1 \\ \\ \left[\begin{array}{c} \underline{x}(t_0) \\ \hline \underline{x}_a(t_0) \end{array}\right] = \left[\begin{array}{c} 0 \\ \hline 0 \end{array}\right] \end{array} \right\}$$

(3.39)

comparing Eqs. (3.38) and (3.39), we note that

$$\underline{F} = F;\ \underline{g} = g;\ \underline{\ell} = k$$

$$Tg = \underline{h}'\sqrt{k} \tag{3.40}$$

$$g'T = \underline{h}\sqrt{k} \tag{3.41}$$

$$\underline{h}'\underline{h} = -\dot{T} + TF + h'h + F'T \tag{3.42}$$

Equations (3.40), (3.41) together with the requirement that $k>0$ imply that T is symmetric and nonnegative definite, and that

$$\underline{h} = \frac{1}{\sqrt{k}}\ g'T \tag{3.43}$$

Furthermore Eqs. (3.42) and (3.43), and the initial conditions of W imply that the nxn matrix T satisfies

$$\dot{T} = -\frac{1}{k}\ Tgg'T + TF + h'h + F'T \tag{3.44}$$

$$T(t_0) = 0$$

Q.E.D.

Hence M and M^a are given by

$$M \longleftrightarrow \begin{cases} \dot{\underline{s}} = F\underline{s} + gu_1 \\ y_1 = \dfrac{1}{\sqrt{k}}\ g'T\underline{s} + \sqrt{k}\ \ u_1 \\ \underline{s}(t_0) = 0 \end{cases} \tag{3.45}$$

$$M^a \longleftrightarrow \begin{cases} \dot{\underline{s}}_a = -F'\underline{s}_a + \frac{1}{\sqrt{k}} \quad Tgu_1 \\ y_1 = -g'\underline{s}_a + \sqrt{k} \quad u_1 \\ \underline{s}_a(t_0) = 0 \end{cases} \tag{3.46}$$

Example 3.1

The following example illustrates the decomposition technique of Theorem 3.3. Let W be given by

$$\dot{x} = -x + u$$
$$y = z$$
$$x(t_0) = 0$$

Let k=1. Therefore T is given by

$$\dot{T} = -T^2 - 2T + 1$$
$$T(t_0) = 0$$

Solving the above Ricatti equation we obtain

$$T = (\sqrt{2}-1) + (1-\sqrt{2})e^{-2\sqrt{2}(t-t_0)}$$

For $t_0 \longrightarrow -\alpha$ we have

$$T = \sqrt{2}-1$$

Thus a state space representation for M in the steady state is

$$\dot{\underline{s}} = -\underline{s} + u_1$$
$$y_1 = (\sqrt{2}-1)\underline{s} + u_1$$
$$\underline{s}(t_0) = 0$$

As a check, we Laplace transform the above equation to obtain the transfer function M(s) of M. Thus

$$M(s) = \frac{s+\sqrt{2}}{s+1}$$

From classical spectral factorization techniques, we have

$$W^a * W + k = \frac{1}{-s+1} \times \frac{1}{s+1} + 1$$

$$= \frac{1}{-s^2+1} + 1 = \frac{-s^2+2}{-s^2+1}$$

$$= \frac{-s+\sqrt{2}}{-s+1} \times \frac{s+\sqrt{2}}{s+1}$$

or

$$M(s) = \frac{s+\sqrt{2}}{s+1}$$

<u>Theorem 3.4</u>

Assume that we are given (a) the system $S_1^a * S_2^a * S_3$, where S_1^a and S_2^a are adjoints of some causal systems S_1 and S_2, and S_3 is a causal system, (b) the following realizations for S_1, S_2 and S_3

$$S_1 \longleftrightarrow (\underset{n\times n}{F_1}, \underset{n\times 1}{g_1}, \underset{1\times n}{h_1}, \underset{1\times 1}{\ell_1}; t_0, 0)$$

$$S_2 \longleftrightarrow (\underset{n\times n}{F_2}, \underset{n\times 1}{g_2}, \underset{1\times n}{h_2}; t_0, 0)$$

$$S_3 \longleftrightarrow (\underset{m\times m}{F_3}, \underset{m\times 1}{g_3}, \underset{1\times m}{h_3}, t_0, 0) ,$$

and (c) the decomposition rule

$$S_1^a * S_2^a * S_3 = P_1^a + P_2$$

where P_1^a is the adjoint of some causal system P_1 and P_2 is causal. Then, P_2 has the realization

$$P_2 \longleftrightarrow (\underset{m\mathrm{x}m}{F_3}, \underset{m\mathrm{x}1}{g_3}, Y; t_0, 0)$$

where

$$Y = \ell_1 g_2' U_1 + g_1' U_2$$

$$\dot{U}_1 = U_1 F_3 + F_2' U_1 + h_2' h_3$$

$$U_1(t_0) = 0$$

$$\dot{U}_2 = U_2 F_3 + F_1' U_2 + h_1' g_2' U_1$$

$$U_2(t_0) = 0$$

Alternatively, Y is given by

$$\begin{bmatrix} \underset{n\mathrm{x}m}{\dot{U}_1} \\ \hdashline \underset{n\mathrm{x}m}{U_2} \end{bmatrix} = \underset{2n\mathrm{x}m}{\dot{U}} = \underset{(2n\mathrm{x}2n)}{\begin{bmatrix} \underset{n\mathrm{x}n}{F_2'} & \vdots & \underset{n\mathrm{x}n}{0} \\ \hdashline \underset{n\mathrm{x}n}{h_1' g_2'} & \vdots & \underset{n\mathrm{x}n}{F_1} \end{bmatrix}} \underset{2n\mathrm{x}m}{U} + \underset{(2n\mathrm{x}m)(m\mathrm{x}m)}{UF_3} + \begin{bmatrix} h_2' h_3 \\ \hdashline 0 \end{bmatrix}$$

$$U(t_0) = 0 \qquad (3.47)$$

$$Y = \left[\ell_1 g_2' \,\vdots\, g_1' \right] U$$

Proof

Let ξ_{1a}, ξ_{2a} and ξ_3 be the state vectors of S_1^a, S_2^a and S_3, respectively. Then $S_1^a * S_2^a * S_3$ has the realization

$$\begin{bmatrix} \dot{\xi}_3 \\ \hline \dot{\xi}_{2a} \\ \hline \dot{\xi}_{1a} \end{bmatrix} = \left[\begin{array}{c|c|c} F_3 & 0 & 0 \\ \hline h_2'h_3 & -F_2' & 0 \\ \hline 0 & -h_1'g_2' & -F_1' \end{array}\right] \begin{bmatrix} \xi_3 \\ \hline \xi_{2a} \\ \hline \xi_{1a} \end{bmatrix} + \begin{bmatrix} g_3 \\ \hline 0 \\ \hline 0 \end{bmatrix} u_1$$

$$y_1 = \left[\begin{array}{c|c|c} 0 & -\ell_1 g_2' & -g_1' \end{array}\right] \begin{bmatrix} \xi_3 \\ \hline \xi_{2a} \\ \hline \xi_{1_a} \end{bmatrix} \qquad (3.48)$$

To decompose the above realization into P_1^a and P_2, we make use of the following equivalence transformation on Eq. (3.48),

$$\underline{U} = \left[\begin{array}{c|c|c} \underset{(m\mathrm{x}m)}{I} & \underset{(m\mathrm{x}n)}{0} & \underset{(m\mathrm{x}n)}{0} \\ \hline \underset{(n\mathrm{x}m)}{U_1} & \underset{(n\mathrm{x}n)}{I} & \underset{(n\mathrm{x}n)}{0} \\ \hline \underset{(n\mathrm{x}m)}{U_2} & \underset{(n\mathrm{x}n)}{0} & \underset{(n\mathrm{x}n)}{I} \end{array}\right]$$

It is easy to verify that

$$\underline{U}^{-1} = \left[\begin{array}{c:c:c} \underset{(m\times m)}{I} & \underset{(m\times n)}{0} & \underset{(m\times n)}{0} \\ \hdashline \underset{(n\times m)}{-U_1} & \underset{(n\times n)}{I} & \underset{(n\times n)}{0} \\ \hdashline \underset{(n\times m)}{-U_2} & \underset{(n\times n)}{0} & \underset{(n\times n)}{I} \end{array}\right]$$

and that

$$\frac{d}{dt}\ \underline{U} = \left[\begin{array}{c:c:c} 0 & 0 & 0 \\ \hdashline \dot{U}_1 & 0 & 0 \\ \hdashline \dot{U}_2 & 0 & 0 \end{array}\right]$$

The results are

$$\left[\begin{array}{c} \dot{\hat{\xi}}_3 \\ \hdashline \dot{\hat{\xi}}_{2a} \\ \hdashline \dot{\hat{\xi}}_{1a} \end{array}\right] = \left[\begin{array}{c:c:c} F_3 & 0 & 0 \\ \hdashline \begin{array}{l} U_1F_3+h_2'h_3 \\ +F_2'U_1-\dot{U}_1 \end{array} & -F_2' & 0 \\ \hdashline \begin{array}{l} U_2F_3+h_1'g_2'U_1 \\ +F_1'U_2-U_2' \end{array} & -h_1'g_2' & -F_1' \end{array}\right] \left[\begin{array}{c} \hat{\xi}_3 \\ \hdashline \hat{\xi}_{2a} \\ \hdashline \hat{\xi}_{1a} \end{array}\right] + \left[\begin{array}{c} g_3 \\ \hdashline 0 \\ \hdashline 0 \end{array}\right] u_1$$

$$y_1 = \left[\begin{array}{c:c:c} \ell_1 g_2' U_1 + g_1' U_2' & -\ell_1 g_2' & -g_1' \end{array}\right] \left[\begin{array}{c} \hat{\xi}_3 \\ \hdashline \hat{\xi}_{2a} \\ \hdashline \hat{\xi}_{1a} \end{array}\right]$$

$$\left[\begin{array}{c} \hat{\xi}_3(t_0) \\ \hdashline \hat{\xi}_{2a}(t_0) \\ \hdashline \hat{\xi}_{1_a}(t_0) \end{array}\right] = \left[\begin{array}{c} 0 \\ \hdashline 0 \\ \hdashline 0 \end{array}\right]$$

The P_2 part of the decomposition can be identified by inspection. Thus

$$P_2 \leftrightarrow \begin{cases} \dot{\hat{\xi}}_3 = F_3\hat{\xi}_3 + g_3u_1 \\ y_1 = (\ell_1 g_2'U_1 + g_1'U_2)\hat{\xi}_3 \end{cases}$$

where U_1 and U_3 are given by

$$\dot{U}_1 = U_1F_3 + F_2'U_1 + h_2'h_3 = 0$$

$$U_1(t_0) = 0$$

and

$$\dot{U}_2 = U_2F_3 + F_1'U_2 + h_1'g_2'U_1$$

$$U_2(t_0) = 0$$

Defining $U = \left[\frac{U_1}{U_2}\right]$, and augmenting the differential equations in U_1 and U_2, we obtain the equation (3.47).

Q.E.D.

<u>Example 3.2</u>

The above decomposition technique is illustrated by the following example. Let S_1^a, S_2^a and S_3 by given by

$$S_1^a \longleftrightarrow (2,1,-1,1;t_0,0)$$

$$S_2^a \longleftrightarrow (1,1,-1;t_0,0)$$

$$S_3 \longleftrightarrow (-2,1,1;t_0,0)$$

such that $S_1^a * S_2^a * S_3$ is given by

$$\begin{bmatrix} \dot{\hat{\psi}} \\ \dot{x}_a \\ \dot{z}_a \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} \hat{\psi} \\ x_a \\ z_a \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u_1$$

$$y_1 = [\, 0 \quad 1 \quad 1 \,] \begin{bmatrix} \hat{\psi} \\ x_a \\ z_a \end{bmatrix}$$

$$\begin{bmatrix} \hat{\psi}(t_0) \\ x_a(t_0) \\ z_a(t_0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

From Theorem 3.4 if $S_1^a * S_2^a * S_3 = P_1^a + P_2$, then P_2 is given by

$$\dot{\underline{a}} = -2\underline{a} + u_1$$

$$y_1 = (U_1 + U_2)\underline{a}$$

where

$$\dot{U}_1 = -3U_1 + 1$$

$$U_1(t_0) = 0$$

and

$$\dot{U}_2 = -4U_2 + U_1$$

$$U_2(t_0) = 0$$

Solving for U_1 and U_2, we obtain

$$U_1(t) = \frac{1}{3} - \frac{1}{3} e^{-3(t-t_0)}$$

and

$$U_2(t) = \frac{1}{12} - \frac{1}{12} e^{-4(t-t_0)} \cdot - \frac{1}{3} e^{-3t} \cdot e^{2t_0}$$

To find the steady state values of U_1 and U_2, we let $t_0 \to -\infty$ and obtain

$$U_1 = \frac{1}{3} \quad ; \quad U_2 = \frac{1}{12}$$

Therefore, in the steady state

$$P_2 \longleftrightarrow \begin{cases} \dot{\underline{a}} = -2\underline{a} + u_1 \\ y_1 = \frac{5}{12} \; \underline{a} \end{cases}$$

or

$$P_2(s) = \frac{5/12}{s+2}$$

As a check, we compute P_2 below via classical factorization techniques. Thus

$$S_1^a * S_2^a * S_3 = \left(\frac{1}{-s+2} + 1\right)\left(\frac{1}{-s+1}\right)\left(\frac{1}{s+2}\right)$$

$$= \left(\frac{-s+s}{-s+2}\right)\left(\frac{1}{-s+1}\right)\left(\frac{1}{s+2}\right)$$

Therefore

$$P_2(s) = \frac{A}{s+2}$$

where

$$A = \left(\frac{-s+3}{-s+2}\right)\left(\frac{1}{-s+1}\right)\Bigg|_{s=-2}$$

$$= \frac{5}{12}$$

or

$$P_2(s) = \frac{5/12}{s+2}$$

Lemma 3.3

The L_2 part of the decomposition given by Eq. (3.34) has the state space representation

$$L_2 \longleftrightarrow \begin{cases} \dot{\underline{a}} = (A-Kc)\underline{a} + Ku_1 \\ y_1 = \frac{1}{\sqrt{k}} g'[U_1 + U_2]\underline{a} \\ \underline{a}(t_0) = 0 \end{cases} \tag{3.49}$$

where U_1 and U_2 are obtained from

$$\dot{U}_1 = F'U_1 + U_1(A-Kc) + h'c \tag{3.50a}$$

$$U_1(t_0) = 0$$

$$\dot{U}_2 = (F' - \frac{1}{\sqrt{k}} Tgg')U_2 + U_2(A-Kc) - Tgg'U_1 \tag{3.50b}$$

$$U_2(t_0) = 0$$

Proof

The proof follows immediately from Theorem 3.4.

Q.E.D.

The next theorem is the key result of this chapter.

Theorem 3.5

The optimal compensator $W_{co} = M^{-1} * L_2$ where M and L_2 are defined by Eqs. (3.31) and (3.34), respectively, has the following realization

$$\begin{bmatrix} \dot{\underline{a}} \\ \hline \dot{\underline{b}} \end{bmatrix} = \left[\begin{array}{c|c} A-Kc & 0 \\ \hline \frac{1}{k}\, gg'Y & F-\frac{1}{\sqrt{k}}\, gg'T \end{array}\right] \begin{bmatrix} \underline{a} \\ \hline \underline{b} \end{bmatrix} + \begin{bmatrix} K \\ \hline 0 \end{bmatrix} Z$$

$$u = \left[\begin{array}{c|c} \frac{1}{k}\, g'Y & -\frac{1}{k}\, g'T \end{array}\right] \begin{bmatrix} \underline{a} \\ \hline \underline{b} \end{bmatrix}$$

Here K and T are defined by Eqs. (2.9) and (3.44) respectively; and Y is an nxm matrix given by

$$Y = U_1 + U_2$$

where

$$\dot{U}_1 = F'U_1 + U_1(A-Kc) + h'c$$

$$U_1(t_0) = 0$$

$$\dot{U}_2 = (F' - \frac{1}{\sqrt{k}}\, Tgg')U_2 + U_2(A-Kc) - Tgg'U_1$$

$$U_2(t_0) = 0$$

Alternatively, Y is given by

$$1 \quad \begin{bmatrix} \dot{U}_1 \\ \hline \dot{U}_2 \end{bmatrix} = \left[\begin{array}{c|c} F' & 0 \\ \hline -Tgg' & F'-\frac{1}{\sqrt{k}}\, Tgg' \end{array}\right] \begin{bmatrix} U_1 \\ \hline U_2 \end{bmatrix} + \begin{bmatrix} U_1 \\ \hline U_2 \end{bmatrix} [A-Kc] + \begin{bmatrix} h'c \\ \hline 0 \end{bmatrix}$$

$$Y = \left[\begin{array}{c|c} I_{nxn} & I_{nxn} \end{array}\right] \begin{bmatrix} U_1 \\ \hline U_2 \end{bmatrix}$$

Proof

The proof follows immediately from Theorem 3.4, Lemma 3.3, and by augmenting the state vectors of M^{-1} and L_2. Theorem 3.5 thus gives an explicit state space realization of the optimal compensator W_{co}.

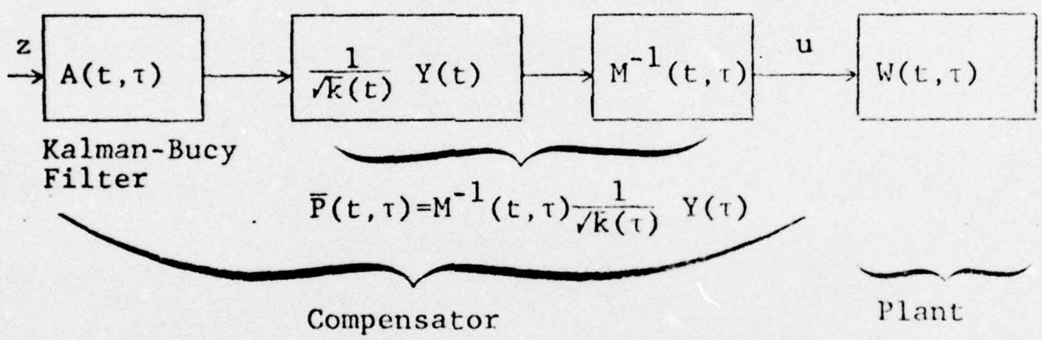
Q.E.D.

3.5 DISCUSSION OF RESULTS

The following comments about the design of the optimal compensator W_{co} are in order. First, from Eq. (3.35) the optimal compensator is a cascaded combination of dynamic systems L_2 and M^{-1}. It is obvious from Eq. (3.49) that L_2 can be viewed as a Kalman-Bucy filter $A(t,\tau)$ for the system DS2 with the output of the filter multiplied by $\frac{1}{\sqrt{k}}\,Y$. In block diagram form, the cascade compensator is shown in Figure (3.2) below. Thus the separation property of the minimum mean-square compensator in the sense of the LQG problem becomes immediately clear. Second, from the structure of the compensator, it is readily apparent that the stability of the Kalman-Bucy filter plays a major role in the stability of the compensator and the overall system. This in turn is tied to the solution of the Riccati equation for the filter error covariance $P(t)$. Furthermore, stability of the Riccati equation (3.44) for T and the differential equation (3.47) for U play the key roles in the stability of the system. More on this topic will be presented in Chapter 5. Third, the presence of the

Lagrange multiplier k, which is always positive, assures the existence of proper inverses $[M^a]^{-1}$ and M^{-1}. The compensator is therefore also always proper. Finally, the compensator equations contain k as a parameter. The choice of k is left to the discretion of the system designer who chooses that value for k which satisfies the constraint

$$E[u^2(t)] \leq \beta$$



Note: $S(t,\tau) = c(t)A(t,\tau)$

OPTIMAL COMPENSATOR STRUCTURE

Figure 3.2

CHAPTER 4

THE CLOSED LOOP PROBLEM

4.1 GENERAL

In Chapter 4, the optimal tracking problem in the closed-loop configuration of Fig. 1.1 is dealt with. Once again, due to the nonstationary nature of the system under study, we formulate and solve the problem in the time domain.

It should be recalled [2-5] that for the optimal tracking problems of the class studied here, the starting point is the formulation of a suitable performance index (PI) in terms of the system parameters. In the open-loop case [2,3,22], such a formulation presents no difficulty. However, the case of the closed-loop problem is entirely different. For example, from Fig. 1.2, a performance index for the closed-loop tracking problem is

$$\begin{aligned} L &= E\left\{[y_d(t) - y(t)]^2\right\} \\ &= E\left\{y_d^2(t) - 2y_d(t)y(t) + y^2(t)\right\} \end{aligned} \tag{4.1}$$

where E is the expectation operator. For simplicity, let us assume that the feedback sensor has unity gain and no memory. Let us proceed to express y(t) in terms of the system parameters. Thus

$$y(t) = \int_{-\alpha}^{+\alpha} W(t,s)\left\{\int_{-\alpha}^{+\alpha} G_c(s,p)[z(p) - y(p)]dp\right\}ds$$

or

$$y(t) + \int_{-\alpha}^{+\alpha} W(t,s) \int_{-\alpha}^{+\alpha} G_c(s,p)\, y(p)\, dp\, ds$$

$$= \int_{-\alpha}^{+\alpha} W(t,s) \int_{-\alpha}^{+\alpha} G_c(s,p)\, z(p)\, dp\, ds \qquad (4.2)$$

It is immediately obvious from Eq. (4.2) that in order to obtain y(t) for further manipulations as part of Eq. (4.1), one needs to solve a double-integral equation in y(t) — a not so easy task! As mentioned above, no such difficulty arises in the open-loop case. This is precisely why the open-loop approach has been used so extensively in the past as an intermediate step towards solving the closed-loop tracking problem.

We overcome the above-mentioned difficulties associated with the closed-loop tracking problem by employing the techniques of the "Algebra of Compositions of Functionals", first developed by Evans and Volterra [18,19] in the course of their studies of integral equations. The Algebra of Compositions is used here to explicitly formulate the system performance index to be minimized in terms of the system parameters. Variational techniques are then used to derive a condition which the optimal compensator must satisfy. This condition turns out to be an integral equation similar to the one derived for the open-loop problem of the previous chapter.

The organization of this chapter is as follows. The next section, Section 4.2 is devoted to the state-space formulation of the closed-loop tracking problem, with which we are concerned. Section 4.3 deals with the "Algebra of Composition" and its application to system theory. In Section 4.4, an integral equation which the optimal closed-loop cascade compensator must satisfy is derived. In Section 4.5, the integral equation is solved and a state space realization of the optimal compensator is given. Finally, in Section 4.6, a brief summary of the results of this chapter and a conclusion are presented.

4.2 PROBLEM FORMULATION

Consider a linear, time-varying, single-input/single-output plant DS1, a reference system DS2 and a feedback sensing system DS3 interconnected in the configuration of Fig. 4.1. The defining equations for the plant DS1 and the reference system DS2 are the same as in Section 3.2, Chapter 3, and are repeated below for convenience.

$$\text{DS1:}\quad \begin{cases} \dot{x}(t) = F(t)x(t) + g(t)u(t) \\ y(t) = h(t)x(t) \\ x(t_0) = 0 \end{cases} \tag{4.3}$$

$$\text{DS2:}\quad \begin{cases} \dot{\psi}(t) = A(t)\psi(t) + b(t)w(t) \\ y_d(t) = c(t)\psi(t) \\ E[\psi(t_0)] = \psi_0 \\ E[\psi(t_0)\psi'(t_0)] = P_0 \end{cases} \tag{4.4}$$

where $x, \psi, F, g, h, u, A, b, c, w$ are exactly as defined previously in Section 3.2.

The input signal to the tracking system is modeled as

$$z(t) = y_d(t) + v(t) \tag{4.5}$$

where $\{v(t),\ t \geq t_0\}$ is a zero-mean, gaussian, white noise process with positive-definite variance R given by

$$E[v(t)v(\tau)] = R(t)\delta(t-\tau),\ T,\tau \geq t_0 \tag{4.6}$$

It is assumed that the variance Q of w defined by

$$E[w(t)w(\tau)] = Q(t)\delta(t-\tau),\ T,\tau \geq t_0 \tag{4.7}$$

is nonnegative definite.

The system output y is sensed by the feedback sensor DS3 which is governed by the equation

$$\text{DS3:} \begin{cases} \dot{\alpha}(t) = \Omega(t)\alpha(t) + \xi(t)y(t) \\ y_0(t) = \gamma(t)\alpha(t) \\ \alpha(t_0) = 0 \end{cases} \tag{4.8}$$

for $t \geq t_0$. Here, α is an ℓ-vector, the state of the feedback sensor, y_0 is a scalar, the feedback signal, and Ω, ξ, and γ are continuous matrix functions of the time, which are $\ell x \ell$, $\ell x 1$ and $1 x \ell$ respectively. The tracking error $y_c(t)$ is defined as

$$y_c(t) = y_d(t) - y(t)$$

For the system performance index L, we choose a suitably weighted sum of the mean-square tracking error and the

mean-square control u. Thus

$$L = E[y_c^2(t)] + k(t)E[u^2(t)] \tag{4.9}$$

where k>0 is a Lagrange multiplier whose value is chosen in the final stage of the design process to satisfy a saturation-like constraint on u of the form

$$E[u^2(t)] \leq \beta > 0 \tag{4.10}$$

The closed-loop tracking problem can now be stated as follows: Given the plants DS1, the reference system DS2, the feedback sensor DS3, the relevant noise statistics and the constraint on u, design an optimal compensator in the closed-loop cascaded configuration of Fig. 4.1 such that the cost fuctional L given by Eq. (4.9) is minimized.

4.3 MATHEMATICAL PRELIMINARIES, ALGEBRA OF COMPOSITION OF FUNCTIONALS [18,19]

Given two functions $f(\gamma,\xi)$ and $g(\gamma,\xi)$ of two variables γ and ξ, the function $h(\gamma,\xi)$ defined by

$$h(\gamma,\xi) = \int_{\xi}^{\gamma} f(\gamma,\alpha)g(\alpha,\xi)d$$

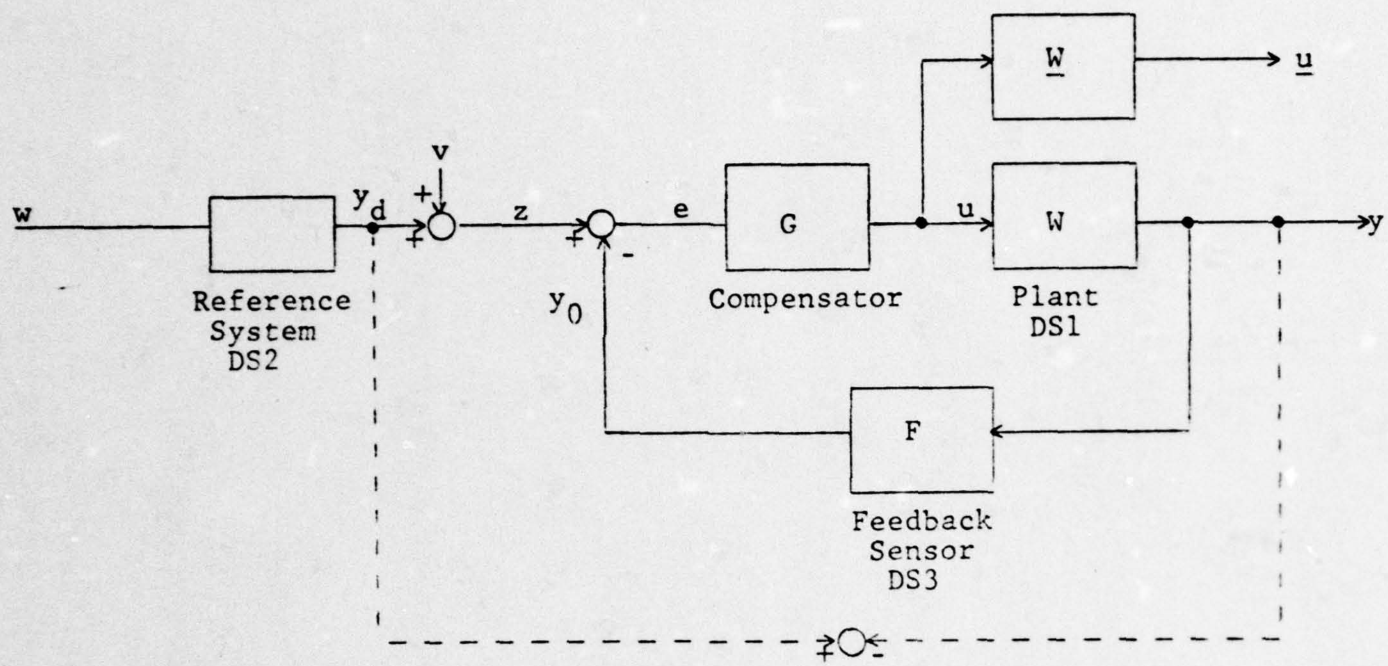
is called product by composition of two functions f and g and is denoted here by

$$\overset{*}{h} = \overset{**}{fg}$$

If

$$\overset{**}{fg} = \overset{**}{gf}$$

then f and g are said to be permutable. The operation of composition is evidently associative and distributive, i.e.,



CLOSED-LOOP TRACKING PROBLEM

Figure 4.1

$$(\overset{*}{f}\overset{*}{g})\overset{*}{h} = \overset{*}{f}(\overset{*}{g}\overset{*}{h})$$

and

$$\overset{*}{f}(\overset{*}{\overline{g+h}}) = \overset{*}{f}\overset{*}{g} + \overset{*}{f}\overset{*}{h}$$

where "+" denotes ordinary addition. Powers of composition for positive integral exponents are defined by

$$\overset{*}{f}\overset{*}{f} = \overset{*}{f}{}^{2};\ \overset{*}{f}{}^{n-1}\overset{*}{f} = \overset{*}{f}{}^{n} = \overset{*}{f}\overset{*}{f}{}^{n-1}$$

Power with exponent zero is defined such that for any f and g

$$\overset{*}{f}{}^{o}\overset{*}{g} = \overset{*}{g}\overset{*}{f}{}^{o},$$

and

$$\overset{*}{f}{}^{o} = \overset{*}{g}{}^{o}$$

Symbolically, then we can define a "unity" function $\overset{*}{1}$ such that

$$\overset{*}{1} = \overset{*}{g}{}^{o} = \overset{*}{f}{}^{o}$$

Therefore for n functions $f_1, f_2, \cdots f_n$ of γ and ξ the following series is well defined.

$$F(f_1, f_2, --- f_n) = \overset{*}{1} + \overset{*}{k}_1\overset{*}{f}{}_1^{a1}\overset{*}{f}{}_2^{a2} --\overset{*}{f}{}_n^{an} + k_2\overset{*}{f}{}_1^{b1}\overset{*}{f}{}_2^{b2}\cdot\cdot\overset{*}{f}{}_n^{bn}$$

$$+\cdots\cdots\cdots\cdots\cdots\infty$$

Here k_is are constants and a_is and b_is are integers. The series is called a functional F of $f_1, f_2, \cdots f_n$.

Fractional powers of composition of functionals are defined in the following manner. If for a given function $h(\gamma, \xi)$, there exists another function $g(\gamma, \xi)$ such that

$$\overset{*}{g}^{n} = \overset{*}{h}$$

where n is a positive integer, then we define

$$\overset{*}{h}^{1/n} = \overset{*}{g}$$

If $\overset{*}{h}^{p} = \overset{*}{g}^{q}$ for some positive integers p and q, then we define

$$\overset{*}{h} = \overset{*}{g}^{q/p} \text{ and } \overset{*}{g} = \overset{*}{h}^{b/q}$$

We say that $\dfrac{\overset{*}{f}}{\overset{*}{g}}$ is a "Fraction of Composition with numerator $\overset{*}{f}$ and denominator $\overset{*}{g}$ if $\overset{*}{f}\overset{*}{g} = \overset{*}{1}$. Two fraction $\dfrac{\overset{*}{f}_1}{\overset{*}{g}_1}$ and $\dfrac{\overset{*}{f}_2}{\overset{*}{g}_2}$ are equal if $\overset{*}{f}_1\overset{*}{g}_2 = \overset{*}{f}_2\overset{*}{g}_1$.

The following relationships are self evident

$$\frac{\overset{*}{g}}{\overset{*}{f}} = \frac{\overset{*}{g}\overset{*}{h}}{\overset{*}{f}\overset{*}{h}} \;;\quad \frac{\overset{*}{h}}{\overset{*}{h}^{o}} = \overset{*}{h};\quad \frac{\overset{*}{g}}{\overset{*}{f}}\,\overset{*}{f} = \overset{*}{g}$$

Negative exponents are defined by

$$\frac{\overset{*}{g}}{\overset{*}{f}} = \overset{*}{g}\overset{*}{f}^{-1}$$

or by

$$\frac{\overset{*}{g}}{\overset{*}{f}} = \overset{*}{f}^{-1}\overset{*}{g}$$

It can be shown that [18] from any analytic function $Z(z_1,z_2,\ldots z_n)$, regular in the region around $z_1=z_2=\ldots.z_n=0$, we can obtain a corresponding functional $F(f_1,f_2\ldots.f_n)$.

Moreover, if $Z(z_1, z_2 \ldots z_n)$ is an infinite series which is convergent when the module z_n are sufficiently small, then the corresponding series $F(f_1, f_2 \ldots f_n)$ is always convergent whatever f_r's may be, provided they are limited. That is, for all f_r's such that

$$|f_r(\gamma,\xi)| < \alpha$$

Furthermore, the algebra for the manipulations of the functionals and their compositions is exactly analogous to ordinary algebra.

Integrals and derivatives of functionals and their compositions are defined in exactly the same fashion as ordinary integrals and derivatives. Thus

$\dfrac{d\overset{*}{F}}{d\overset{*}{f}}(f_1, f_2, \text{--} f_n)$ is itself a new function by composition of $\overset{*}{F}$.

Linear Systems

The compositional algebra presented above is directly applicable to linear systems. Thus two linear systems with impulse response functions $f(t,\tau)$ and $g(t,\tau)$, when cascaded, are equivalent to a system given by the impulse response

$$h(t,\tau) = \int_{-\alpha}^{t} f(t,\sigma) g(\sigma,\tau) d\sigma$$

Or, alternately, we can write

$$\overset{*}{h} = \overset{*}{f}\overset{*}{g}$$

Thus signals in a particular part of an interconnected linear system can be represented as functionals and their compositions, operating on signals existing in some other part of the system. The fact that these functionals and their compositions can be manipulated just like functions in ordinary algebra and calculus, is of great value in optimization problems.

4.4 THE INTEGRAL EQUATION

In this section, we derive a necessary and sufficient condition which the closed-loop optimal cascade compensator must satisfy.

In order to keep track of the variables, as it was done in the last chapter, here too we choose to constrain a *functional* $\underline{u}$ *of* u *rather than* u. *Thus the revised cost* functional L' is

$$L' = E[y_d^2 - 2y_d y + y^2] + kE[\underline{u}^2] \tag{4.11}$$

where

$$\underline{u} = \int_{-\infty}^{+\infty} \underline{W}(t,\tau)u(\tau)d\tau, \quad t>t_0$$

From Fig. 4.1

$$e = z - \overset{*}{F}\overset{*}{W}\overset{*}{G}e$$

or

$$(\overset{*}{I} + \overset{*}{F}\overset{*}{W}\overset{*}{G})e = z$$

or

$$e = (\overset{*}{I} + \overset{*}{F}\overset{*}{W}\overset{*}{G})^{-1}z$$

Thus

$$y = [\overset{*}{W}\overset{*}{G}(\overset{*}{I} + \overset{*}{F}\overset{*}{W}\overset{*}{G})^{-1}]z \tag{4.12}$$

and

$$\underline{u} = [\underline{\overset{*}{W}}\overset{*}{G}(\overset{*}{I} + \overset{*}{F}\overset{*}{W}\overset{*}{G})^{-1}]z \tag{4.13}$$

For convenience, in what follows, we omit the "asterisk". Substituting Eqs. (4.12) and (4.13) into Eq. (4.11), we obtain

$$L' = E\Bigg[y_d^2 - 2y_d\Big\{[WG(1+FWG)^{-1}]z\Big\} + \Big\{[WG(1+FWG)^{-1}]z\Big\}^2 + k\Big\{[\underline{W}G(1+FWG)^{-1}]z\Big\}^2\Bigg]$$

Applying the calculus of variations, we let

$$G = G_c + eG_e$$

where G_c is the value of G which makes L' stationary, and eG_e is the variation of G_c. Thus

$$\begin{aligned} L' = E\Bigg[& y_d^2 - 2y_d\Big\{\Big[W[G+eG_e](1+FW[G_c+eG_e])^{-1}\Big]z\Big\} \\ & + \Big\{\Big[W[G_c+eG_e](1+FW[G_c+eG_e])^{-1}\Big]z\Big\}^2 \\ & + k\Big\{\Big[\underline{W}[G_c+eG_e](1+FW[G_c+eG_e])^{-1}\Big]z\Big\}^2\Bigg] \end{aligned} \tag{4.14}$$

The stationary value of g_c is next found by setting

$$\left.\frac{dL'}{de}\right|_{e=0} = 0$$

Differentiating the right-hand-side of Eq. (4.14) term by term with respect to e, we have

First Term

$$\frac{d}{dE}\; E[y_d^2] = 0$$

Second Term

$$\frac{d}{dE}\, E\left[-2y_d\left\{\left[W[G_c+eG_e](1+FW[G_c+eG_e])^{-1}\right]z\right\}\right]_{e=0}$$

$$= E\left[-2y_d\left\{\left[WG_c\;\frac{d}{de}(1+FW[G_c+eG_e])^{-1}\right.\right.\right.$$

$$\left.\left.\left.+\;\frac{d}{dE}\,W[G_c+eG_e]x(1+FWG_c)^{-1}\right]z\right\}\right]_{=0}$$

$$= E\left[-2y_d\left\{\left[-WG_c(1+FWG_c)^{-2}FWG_e+WG_e(1+FWG_c)^{-1}\right]z\right\}\right]$$

$$= E\left[-2y_d\left\{\left[-WG_cFWG_e(1+FWG_c)^{-2}+WG_e(1+FWG_c)^{-1}\right]z\right\}\right]$$

Third Term

$$\frac{d}{dE}\, E\left[\left\{\left[W[G_c+eG_e](1+FW[G_c+eG_e])^{-1}\right]z\right\}^2\right]_{=0}$$

$$= 2E\left[\left\{\left[WG_c(1+FWG_c)^{-1}\right]z\right\}x\right.$$

$$\left.\frac{d}{dE}\left\{\left[W[G_c+eG_e](1+FW[G_c+eG_e])^{-1}\right]z\right\}_{e=0}\right]$$

$$= 2E\left[\left\{\left[WG_c(1+FWG_c)^{-1}\right]z\right\}x\right.$$

$$\left\{\left[-WG_cFWG_e(1+FWG_c)^{-1}+WG_e(1+FWG_c)^{-1}\right]\right\}z\Big]$$

<u>Fourth Term</u>

$$k\,\frac{d}{de}\,E\left[\left\{\left[\underline{W}[G_c+eG_e](1+FW[G_c+eG_e])^{-1}\right]z\right\}^2\right]_{=0}$$

$$= 2k\;E\left[\left\{\left[\underline{W}G_c(1+FWG_c)^{-1}\right]z\right\}x\right.$$

$$\frac{d}{de}\;\left\{\left[\underline{W}[G_c+eG_e](1+FW[G_c+eG_e])^{-1}\right]z\right\}\Big]_{=0}$$

$$= 2k\;E\left[\left\{\left[\underline{W}G_c(1+FWG_c)^{-1}\right]z\right\}x\right.$$

$$\left.\left\{\left[-\underline{W}G_cFWG_e(1+FWG_c)^{-2}+WG_e(1+FWG_c)^{-1}\right]z\right\}\right]$$

We post-multiply each of the above terms by the operator $(1+FWG_c)^2$, gather all four terms and obtain

$$E\left[-2y_d\left\{-WG_cFWG_e+WG_e(1+FWG_c)\right\}z\right]$$

$$+\;2E\left[\left\{\left[WG_c(1+FWG_c)^{-1}\right]z\right\}\left\{\left[-WG_cFWG_e+WG_e(1+FWG_c)\right]z\right\}\right]$$

$$+\;2kE\left[\left\{\left[\underline{W}G_c(1+FWG_c)^{-1}\right]z\right\}\left\{\left[-\underline{W}G_cFWG_e+WG_e(1+FWG_e)^{-1}\right]z\right\}\right]$$

$$= 0$$

Post-multiplying each term of the above equation by the operator $(1+FWG_c)$ and simplifying, we obtain

$$- 2E\left[\left\{[1+FWG_c]y_d\right\}\left\{[WG_e]z\right\}\right]$$

$$+ 2E\left[\left\{[WG_c]z\right\}\left\{[WG_e]z\right\}\right]$$

$$+ 2kE\left[\left\{[\underline{W}_c]z\right\}\left\{[WG_e]z\right\}\right] = 0 \qquad (4.15)$$

Since $G_e \equiv 0$ for $t<t_0$ and arbitrary for $t\geq t_0$, Eq. (4.15) can be rewritten as

$$E\left[\left\{[WG_c]z\right\}\left\{[W]z\right\}\right] + kE\left[\left\{[\underline{W}G_c]z\right\}\left\{[\underline{W}]z\right\}\right]$$

$$= E\left[\left\{y_d\right\}\left\{[W]z\right\}\right]$$

$$+ E\left[\left\{[FWG_c]y_d\right\}\left\{[W]z\right\}\right] \qquad (4.16)$$

for $t\geq t_0$.

It should be noted that

$$[WG_c]z = \int_{-\infty}^{+\infty} d\tau_2 \int_{-\infty}^{+\infty} d\tau_1 \; W(t,\tau_2)G_c(\tau_2,\tau_1)z(\tau_1)$$

$$[\underline{W}G_c]z = \int_{-\infty}^{+\infty} d\tau_2 \int_{-\infty}^{+\infty} d\tau_1 \; \underline{W}(t,\tau_2)G_c(\tau_2,\tau_1)z(\tau_1)$$

$$[W]z = \int_{-\infty}^{+\infty} d\sigma \; W(t,\sigma)z(\sigma)$$

$$[\underline{W}]z = \int_{-\infty}^{+\infty} d\sigma_1 \; \underline{W}(t,\sigma)z(\sigma)$$

$$[FWG_c]y_d = \int_{-\infty}^{+\infty} d\tau_3 \int_{-\infty}^{+\infty} d\tau_2 \int_{-\infty}^{+\infty} d\tau_1 \; F(t,\tau_3)W(\tau_3,\tau_2)G_c(\tau_2,\tau_1)y_d(\tau_1)$$

where $t \geq \tau_3 \geq \tau_2 \geq \tau_1 \geq t_0$, and $t \geq \sigma \geq t_0$. Substituting the integral representation for the operator representation in Eq. (4.16), and interchanging the order of integral operators with the expectation operator, we obtain the following integral equation.

$$\int_{-\infty}^{+\infty} d\tau_2 \int_{-\infty}^{+\infty} d\tau_1 \int_{-\infty}^{+\infty} d\sigma \; W(t,\tau_2)G_e(\tau_2,\tau_1)W(t,\sigma)E[z(\tau_1)z(\sigma)]$$

$$+ \; k \int_{-\infty}^{+\infty} d\tau_2 \int_{-\infty}^{+\infty} d\tau_1 \int_{-\infty}^{+\infty} d\sigma \; \underline{W}(t,\tau_2)G_c(\tau_2,\tau_1)\underline{W}(t,\sigma)E[z(\tau_1)z(\sigma)]$$

$$= \int_{-\infty}^{+\infty} d\sigma_1 \; W(t,\sigma)E[y_d(t)z(\sigma)]$$

$$+ \int_{-\infty}^{+\infty} d\tau_3 \int_{-\infty}^{+\infty} d\tau_2 \int_{-\infty}^{+\infty} d\tau_1 \int_{-\infty}^{+\infty} d\sigma \; F(t,\tau_3)W(\tau_3,\tau_2)G_e(\tau_2,\tau_1)W(t,\sigma)$$

$$E[y_d(\tau_1)z(\sigma)] \qquad (4.17)$$

for $t \geq \tau_3 \geq \tau_2 \geq \tau_1 \geq t_0$, and $t \geq \sigma \geq t_0$.

The key result of this section is given by Theorem 4.1 below.

Theorem 4.1

The necessary and sufficient condition for the physically realizable closed-loop cascade compensator G_{co} to minimize L' is that it satisfy the integral equation (4.17).

Proof

The necessary part of the proof is immediate from the above derivation. The sufficiency follows from consideration of the second variation.

Comment

In Eq. (4.17), if we set $F(t,\tau)\equiv 0$ (which is equivalent to breaking the feedback loop), the equation reduces to the integral equation (3.17) for the open-loop problem. This is, of course, what one should expect.

4.5 THE OPTIMAL COMPENSATOR

The optimal compensator $G_c(t,\tau)$ is obtained by solving the above integral equation in the time domain. The solution of Eq. (4.17) follows along the lines of the solution of Eq. (3.17). It should be noted that the first three terms of Eq. (4.17) are identical with the three terms of Eq. (3.17). Therefore, the results of Lemma 3.2, and Theorems 3.1 and 3.2 are directly applicable here. The following theorem summarizes the steps necessary to convert Eq. (4.17) involving covariances into an equation involving operators.

Theorem 4.2

The integral equation (4.17) and the following operator equation are equivalent

$$W^a * W * G_e + k\, \underline{W}^a * \underline{W} * G_e$$
$$= W^a * S + S^a * S * F * W * G_c \qquad (4.18)$$

for $t \geq t_0$, where W^a and $\underline{W}^a$ are, respectively, the noncausal adjoints of W and $\underline{W}$; and S is the Kalman-Bucy filler associated with the process $z(\cdot)$, and is given by Eq. (2.8).

Proof

The proof follows immediately if we substitute for $z(t)$ in Eq. (4.17), the integral

$$\int_{-\infty}^{+\infty} W(t,\tau)\upsilon(\tau)d\tau, \quad t \leq \tau \leq t_0$$

where $\upsilon(\cdot)$ is the innovation process associated with $z(\cdot)$, and use the arguments developed in Lemmas 3.1, 3.2 and Theorem 3.2.

Q.E.D.

Since Eq. (4.18) holds for only $t \geq t_0$, and involves linear operators and their adjoints, it can be solved via Wiener's spectral factorization techniques. Once again, as was done in the last chapter, we set $\underline{W}=\delta(t-\tau)$ and obtain the following equation

$$(W^a * W + k) * G_c = W^a * S[1 + F * W * G_c]$$

or

$$(W^a * W + k) * G_c * [1 + F * W * G_c]^{-1} = W^a * S$$

Let

$$W^a * W + k = M^a * M$$

where M^a is the noncausal adjoint of M. Thus

$$M * G_c * [1 + F * W * G_c]^{-1} = [M^a]^{-1} * W^a * S$$

where $[M^a]^{-1}$ is the inverse of M^a in the sense of Theorem 2.2.

Furthermore, let

$$[M^a]^{-1} * W^a * S = \underline{L}^a + L_2$$

where $\underline{L}$ is a causal operator and L_2 is noncausal. Then

$$M * G_c [1 + F * W * G_c]^{-1} = L_2$$

or

$$G_c = M^{-1} * L_2 * [1 + F * W * G_c]$$

or

$$G_c - M^{-1} * L_2 * F * W * G_{co} = M^{-1} * L_2$$

or

$$(1 - M^{-1} * L_2 * F * W) * G_{co} = M^{-1} * L$$

or

$$G_c = [1 - M^{-1} * L_2 * F * W]^{-1} * M^{-1} * L_2 \qquad (4.19)$$

Equation (4.19) gives the impulse response of the closed-loop optimal compensator in terms of the operators M, L_2, F and W. Once again if we set $F(t,\tau)\equiv 0$, Eq. (4.19) reduces to Eq. (3.35), the equation for the open-loop compensator of the last chapter. The state-space realization of Eq. (4.19) is given by Theorem 4.3 below.

<u>Theorem 4.3</u>

A realization of the closed-loop optimal compensator G_c is given by

$$
\begin{bmatrix} \dot{\underline{a}} \\ \dot{\underline{b}} \\ \dot{\hat{x}} \\ \dot{\hat{\alpha}} \\ \dot{\hat{a}} \\ \dot{\hat{b}} \end{bmatrix}
=
\left[\begin{array}{c|c|c|c|c|c}
A-Kc & 0 & 0 & 0 & 0 & 0 \\ \hline
\begin{matrix} F \\ \frac{1}{k}gg'Y \end{matrix} & \begin{matrix} F \\ -\frac{1}{\sqrt{k}}gg'T \end{matrix} & 0 & 0 & 0 & 0 \\ \hline
\frac{1}{k}gg'Y & -\frac{1}{k}gg'T & F & 0 & \frac{1}{k}gg'Y & \frac{1}{k}gg'T \\ \hline
0 & 0 & \xi h & K\gamma & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & A-Kc & 0 \\ \hline
0 & 0 & 0 & 0 & \frac{1}{k}gg'Y & \begin{matrix} F \\ -\frac{1}{\sqrt{k}}gg'T \end{matrix}
\end{array}\right]
\begin{bmatrix} \underline{a} \\ \underline{b} \\ \hat{x} \\ \hat{\alpha} \\ \hat{a} \\ \hat{b} \end{bmatrix}
$$

$$
+ \begin{bmatrix} K \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} z
$$

$$u = \left[\begin{array}{c|c|c|c|c|c} 0 & 0 & 0 & 0 & \frac{1}{k} g'Y & -\frac{1}{k} g'T \end{array} \right] \begin{bmatrix} \underline{a} \\ \underline{b} \\ \hat{x} \\ \hat{\alpha} \\ \hat{a} \\ \hat{b} \end{bmatrix}$$

where K, T and Y are given by Eqs. (2.9), (3.36) and (3.47) respectively.

Proof

From Chapter 3, $M^{-1} * L_2$ has the realization given by

$$\begin{bmatrix} \dot{\underline{a}} \\ \hline \underline{b} \end{bmatrix} = \left[\begin{array}{c|c} (A-Kc) & 0 \\ \hline \frac{1}{k} gg'Y & F - \frac{1}{\sqrt{k}} gg'T \end{array} \right] \begin{bmatrix} \underline{a} \\ \hline \underline{b} \end{bmatrix} + \begin{bmatrix} K \\ \hline 0 \end{bmatrix} u_1$$

$$y_1 = \left[\begin{array}{c|c} \frac{1}{k} g'Y & -\frac{1}{k} g'T \end{array} \right] \begin{bmatrix} \underline{a} \\ \hline \underline{b} \end{bmatrix} \qquad (4.20)$$

where K and T satisfy the Riccati equations (2.9) and (3.36) respectively, and Y is given by Eq. (3.47). The realization of $1-M^{-1} * L_2 * F * W$ is thus given by

$$
\begin{bmatrix} \dot{x} \\ \dot{\alpha} \\ \dot{\underline{a}} \\ \dot{\underline{b}} \end{bmatrix} =
\left[\begin{array}{c|c|c|c} F & 0 & 0 & 0 \\ \hline \xi h & \Omega & 0 & 0 \\ \hline 0 & K & A-Kc & 0 \\ \hline 0 & 0 & \frac{1}{k} gg'Y & F - \frac{1}{\sqrt{k}} gg'T \end{array}\right]
\begin{bmatrix} x \\ \alpha \\ \underline{a} \\ \underline{b} \end{bmatrix} +
\begin{bmatrix} g \\ 0 \\ 0 \\ 0 \end{bmatrix} u_1
$$

$$
y_1 = \left[\begin{array}{c|c|c|c} 0 & 0 & -\frac{1}{k} g'Y & +\frac{1}{k} g'T \end{array}\right]
\begin{bmatrix} x \\ \alpha \\ \underline{a} \\ \underline{b} \end{bmatrix} + u_1
$$

Thus $(1-M^{-1} * L_2 * F * W)^{-1}$ from [21] has the state-space representation

$$
\begin{bmatrix} \dot{\hat{x}} \\ \dot{\hat{\alpha}} \\ \dot{\hat{\underline{a}}} \\ \dot{\hat{\underline{b}}} \end{bmatrix} =
\left[\begin{array}{c|c|c|c} F & 0 & \frac{1}{k} gg'Y & \frac{1}{k} gg'T \\ \hline \xi h & \Omega & 0 & 0 \\ \hline 0 & K & A-Kc & 0 \\ \hline 0 & 0 & \frac{1}{k} gg'Y & F - \frac{1}{\sqrt{k}} gg'T \end{array}\right]
\begin{bmatrix} \hat{x} \\ \hat{\alpha} \\ \hat{\underline{a}} \\ \hat{\underline{b}} \end{bmatrix} +
\begin{bmatrix} g \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$y_1 = \left[\begin{array}{c|c|c|c} 0 & 0 & \frac{1}{k}g'Y & -\frac{1}{k}g'T \end{array} \right] \begin{bmatrix} \hat{x} \\ \hat{\alpha} \\ \hat{\underline{a}} \\ \hat{\underline{b}} \end{bmatrix} + u_1 \qquad (4.2\ )$$

Theorem 4.3 will follow immediately if we augment Eq. (4.20) with the above equation.

Q.E.D.

Theorem 4.3 gives a state-space realization of the optimal closed-loop cascade compensator. The realization is explicit and can be easily computed from a knowledge of the system parameters. Here also, the compensator equations contain k as a parameter. The value of k is chosen by the designer via trial and error or graphical techniques such that the constraint equation (4.10) is satisfied.

4.6 SUMMARY AND CONCLUSION

In this chapter, the closed-loop cascade compensator design problem for nonstationary systems was formulated and solved. The approach used was a time-domain one, and the final design of the compensator was given in terms of its state-space realization.

The usefulness of the closed-loop design versus the open-loop design of the last chapter lies in the fact that, in general, closed-loop systems are less sensitive to

system disturbances and variations in system parameters. But before such a system is implemented in practice, one has to be assured of the stability of the loop. This important question, that of the stability of the closed-loop as well as the open-loop system, is the topic of the next chapter of this dissertation.

CHAPTER 5

STABILITY

5.1 GENERAL

In Chapter 5 we derive the conditions for the stability of the open-loop and the closed-loop compensators, as well as the resulting systems.

It was pointed out in Chapter 3 that the stability properties of the Kalman-Bucy filter associated with the input signal $z(\cdot)$ play a major role in the stability of the open-loop system. It will be shown in the sequel that the same holds true for the closed-loop system as well.

This chapter is organized into five sections. In the next section, Section 5.2, we present a brief introduction to the stability theory, and state conditions for the stability of Kalman-Bucy filters. Sections 5.3 and 5.4 deal, respectively, with the stability of the open-loop and the closed-loop systems. Finally, in Section 5.5, a brief summary of our results and a conclusion are presented.

5.2 STABILITY AND THE KALMAN-BUCY FILTER

We start with a few basic theorems of the stability theory. For the definition of the terms used here, the reader is referred to [10] and [24]. Consider a linear, dynamic autonomous system

$$\dot{x}(t) = A(t)x(t) \tag{5.1}$$

Theorem 5.1 [24]

The zero state of Eq. (5.1) is asymptotically stable if there exist positive numbers k_1 and k_2 such that

$$||\Phi(t,t_0)|| < k_1 e^{-k_2(t-t_0)}$$

for any t_0 and for all $t \geq t_0$, where $||\cdot||$ denotes the Euclidean norm. Alternatively, the zero state of Eq. (5.1) is asymtotically stable if

$$||\Phi(t,t_0)|| \to 0 \text{ as } t \to \infty$$

Note that for linear system, asymptotic stability also implies exponential stability.

Next, consider the linear dynamic system E given by

$$E: \quad \dot{x}(t) = A(t)x(t) + B(t)u(t) \qquad (5.2a)$$

$$y(t) = C(t)x(t) \qquad (5.2b)$$

The bounded-input bounded-output (BIBO) stability [24] of E requires that

$$\int_{t_0}^{t} ||C(t)\Phi(t,\tau)B(\tau)|| d\tau < k < \infty$$

for any t and all $t > t_0$.

We will be concerned here with the total stability or T-stability of systems [24] which requires that for any initial state, and for any bounded input, the output as well as the state variables of the system E are bounded. The following theorem gives the conditions for the T-stability of E.

Theorem 5.2 [24]

If the matrices B and C in Eq. (5.2) are bounded on $(-\infty,\infty)$, and the zero-state of $\dot{x}=Ax$ is asymptotically stable, the system E is T-stable.

The following two theorems state the conditions for the boundedness of the solutions of matrix Riccati equations, and the stability of Kalman-Bucy filters.

Theorem 5.3 [10]

Consider the linear dynamic system Fl given by

$$\text{Fl:}\quad \begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ y(t) &= C(t)x(t) \end{aligned}$$

and the matrix Riccati equation

$$\begin{aligned} \dot{P}(t) = A(t)P(t) + P(t)A'(t) &+ B(t)K_1(t)B'(t) \\ &- P(t)C'(t)K_2(t)C(t)P(t) \end{aligned} \qquad (5.3)$$

Then if (a) A(t) is continuous and bounded, (b) C(t), B(t), $K_1(t)$ and $K_2(t)$ are piecewise continuous and bounded, and furthermore, that K_1 and K_2 are positive definite, (c) the homogeneous system $\dot{x}=Ax$ is exponentially stable, the solution P(t) of the Riccati equation (5.3) with the initial condition $P(t_0)=P_0>0$ is bounded, and converges to a non-negative definite matrix $\bar{P}(t)$ as $t_0\to-\infty$. $\bar{P}(t)$ is a solution Eq. (5.3).

Theorem 5.4 [10]

Consider the measurement process $z(\cdot)$ of Chapter 2 and its associated linear model, whose governing equations

(2.3) and (2.4) are rewritten below

$$z(t) = y_d(t) + v(t) \tag{2.3}$$

$$\begin{aligned} \dot{\psi}(t) &= A(t)\psi(t) + B(t)w(t) \\ y_d(t) &= C(t)\psi(t) \end{aligned} \tag{2.4}$$

Then if (a) $A(t)$ is continuous and bounded, (b) $B(t)$, $C(t)$, $Q(t)$ and $R(t)$ are piecewise continuous and bounded, and furthermore, that $Q(t)$ and $R(t)$ are positive definite, and (c) the system given by Eq. (2.4) is either

I. both uniformly completely reconstructible, and uniformly completely controllable, or

II. asymptotically stable.

the Kalman-Bucy filter given by Eqs. (2.8) and (2.9) is asymptotically stable.

5.3 THE OPEN-LOOP SYSTEM

In this section we derive the conditions for the T-stability of the open-loop compensator and the overall system.

Recall from Chapter 3 that the impulse response $W_{co}(t,\tau)$ of the open-loop compensator is given by

$$W_{co} = M^{-1} * L_2$$

Note from [24] that the T-stability of two or more tandem-connected linear systems guarantees the T-stability of the resulting overall system. Therefore for W_{co} to be T-stable it is sufficient that M^{-1} and L_2 be each T-stable. First we consider M^{-1}.

From Chapter 3, the state space representation of M^{-1} is

$$M^{-1} \leftrightarrow \begin{cases} \dot{\underline{b}} = (F - \frac{1}{k} gg'T)\underline{b} + \frac{1}{\sqrt{k}} g\, u_1 \\ y_1 = -\frac{1}{k} g'T\, \underline{b} + \frac{1}{\sqrt{k}} u_1 \end{cases} \tag{5.4}$$

where, T(t) satisfies the matrix Riccati equation

$$\dot{T} = TF + F'T + h'h - Tg\,\frac{1}{k}\,g'T \tag{5.5}$$
$$T(t_0) = 0$$

Obviously, the boundedness of T(t) is a necessary condition for the T-stability of M^{-1}. The following theorem establishes the conditions for the boundedness of T(t).

Theorem 5.5

Consider the dynamic system DS1 given by Eq. (3.1) and the Riccati equation (5.5). If (a) F(t) is continuous and bounded, (b) g(t), h(t), and k(t) are piecewise continuous and bounded, (c) k(t) is positive definite, such that $\frac{1}{k}$ exists for all $t \geq t_0$, and (d) the zero-state of the homogeneous system $\dot{x}=Fx$ is asymptotically stable, the solution T(t) of the matrix Riccati equation (5.4) is bounded.

Proof

The proof of Theorem 5.5 is an immediate consequence of Theorem 5.3.

Next, we prove in Theorem 5.6 that the conditions under which T(t) is bounded are also sufficient to ensure the T-stability of M^{-1}. For the proof of this and a

subsequent theorem, we need the following lemma.

Lemma 5.1

If F is a negative definite matrix, the homogeneous system $\dot{x}=Fx$ is asymptotically stable.

Proof

Choose a positive definite quadratic form

$$V = x'Ix$$

Therefore

$$\dot{V} = 2x'\dot{x} = 2x'Fx$$

For negative definite F, $\dot{V}$ is negative definite, and therefore, from Lyapanov's stability criterion [29], $\dot{x}=Fx$ is asymptotically stable.

Q.E.D.

Theorem 5.6

Under the assumptions (a) through (d) of Theorem 5.5, the system M^{-1} given by Eq. (5.4) is T-stable.

Proof

First consider the homogeneous system

$$\dot{\underline{b}} = (F - \frac{1}{k} gg'T)\underline{b}$$

The state transition matrix $\underline{S}(t,\tau)$ of M^{-1} is given by

$$\underline{S}(t,\tau) = \exp\left[\int_{\tau}^{t} (F - \frac{1}{k} gg'T)d\sigma\right]$$

$$= \exp\left[\int_{\tau}^{t} F\, d\sigma\right] \times \exp\left[\int_{\tau}^{t} -\frac{1}{k} gg'T\, d\sigma\right]$$

$$= \Phi_1(t,\tau)\Phi_2(t,\tau)$$

where $\Phi_1(t,\tau)$ and $\Phi_2(t,\tau)$ are the state transition matrices associated with F and $-\frac{1}{k}gg'T$ respectively. Denoting the Euclidian norm by $||\cdot||$, we have

$$||\mathcal{S}(t,\tau)|| \le ||\Phi_1(t,\tau)||\ ||\Phi_2(t,\tau)|| \qquad (5.6)$$

From the assumptions and the consequence of Theorem 5.5, $\frac{1}{k}gg'T$ is bounded and positive semidefinite, and therefore $-\frac{1}{k}gg'T$ is bounded and negative semidefinite. This implies, from [8] that

$$||\Phi_2(t,\tau)|| < K_1 < \alpha$$

Furthermore, since x=Fx is asymptotically stable, Theorem 5.1 implies that

$$||\Phi_1(t,\tau)|| \to 0 \text{ as } t \to \alpha$$

and therefore

$$||\mathcal{S}(t,\tau)|| \to 0 \text{ as } t \to \alpha$$

Theorem 5.5 also implies that $\frac{1}{\sqrt{k}}g$ and $\frac{1}{k}g'T$ are bounded. Therefore M^{-1} is T-stable.

Q.E.D.

Finally, we proceed to establish conditions for the T-stability of L_2. Recall from Eq. (3.49) that the boundedness of the nxm matrices U_1 and U_2 is a necessary condition for the T-stability of L_2. Here U_1 and U_2 are

solutions of the first order linear matrix differential equations (3.50a) and (3.50b) respectively. The following Lemma 5.2 which establishes conditions for the boundedness of the solutions of linear matrix equations will be required in the subsequent discussion.

Lemma 5.2

Consider the matrix differential equation

$$\dot{X}(t) = A(t)X(t) + X(t)B(t) + C(t) \qquad (5.7)$$

$$X(t_0) = D_0$$

Here X, A, B, C and D_0 denote, respectively, matrices of dimensions nxm, nxn, mxm, nxm and nxm. The elements of A, B and C are bounded, piecewise continuous time functions, and D_0 is a matrix of constant coefficients. If both A and B are stability matrices [25], that is, both

$$\dot{X}_1 = A(t)X(t)$$

and

$$\dot{X}_2 = B'(t)X_2(t)$$

are asymptotically stable, the solution X(t) of Eq. (5.7) is bounded.

Proof

Let $\Phi_1(t,\tau)$ and $\Phi_2(t,\tau)$ be the state transition matrices of the two matrix differential systems

$$\dot{P}_1 = A\,P_1$$

and

$$\dot{P}_2 = P_2\,B$$

such that

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$$\frac{d}{dt}\hat{\Phi}_1(t,\tau) = A\hat{\Phi}_1(t,\tau);\ \hat{\Phi}_1(t,t) = I_{nxn}$$

$$\frac{d}{dt}\hat{\Phi}_2(t,\tau) = B'\hat{\Phi}_2(t,\tau);\ \hat{\Phi}_2(t,t) = I_{mxm}$$

Then from [26], the solution X(t) of Eq. (5.7) is given by

$$X(t) = \int_{t_0}^{t} \hat{\Phi}_1(t,\tau)C(\tau)\hat{\Phi}_2(\tau,t)d\tau$$

Thus

$$||X(t)|| \leq \int_{t_0}^{t} ||\hat{\Phi}_1(t,\tau)C(\tau)||\,||\hat{\Phi}_2(\tau,t)||d\tau$$

Since B is a stability matrix, from a theorem in [8], $\hat{\Phi}_2(\tau,t)$ is bounded. That is

$$||\hat{\Phi}_1(t,\tau)|| \leq \hat{K}_1 < \alpha$$

and therefore,

$$||X(t)|| \leq \hat{K}_1 \int_{t_0}^{t} ||\hat{\Phi}_1(t,\tau)C(\tau)||d\tau$$

Furthermore, asymptotic stability of $\dot{P}_1 = AP_1$ implies that

$$\int_{t_0}^{t} ||\hat{\Phi}_1(t,\tau)C(\tau)||d\tau \leq \hat{K}_2 < \alpha$$

Thus $||X(t)|| \leq \hat{K}_1\hat{K}_2 < \alpha$, and hence X(t) is bounded.

Q.E.D.

Using the results of the above lemma, we state and prove in Theorem 5.7 below the stability conditions for L_2.

Theorem 5.7

Consider the dynamic systems DS1 and DS2 given by Eqs. (3.1) and (3.2) respectively. Under the assumptions (a) through (d) of Theorem 5.5 concerning the parameters of DS1, and the following additional assumptions (e) through (h) concerning the parameters of DS2; (e) $A(t)$ is continuous and bounded, (f) $b(t)$, $C(t)$, $Q(t)$ and $R(t)$ are piecewise continuous and bounded, (g) Q and R are positive definite, (h) the dynamic system DS2 is either

I. both uniformly completely reconstructible and uniformly completely controllable, or

II. asymptotically stable

the dynamic system L_2 given by Eqs. (3.49) and (3.50) below is T-stable.

$$L_2 \leftrightarrow \begin{cases} \dot{\underline{a}} = (A-Kc)\underline{a} + Ku_1 \\ y_1 = \frac{1}{\sqrt{k}}\, g'[U_1+U_2]\underline{a} \\ \underline{a}(t_0) = 0 \end{cases} \tag{3.49}$$

where

$$\dot{U}_1 = F'U_1 + U_1(A-Kc) + h'c \tag{3.50a}$$

$$U_1(t_0) = 0$$

$$\dot{U}_2 = (F'-\frac{1}{\sqrt{k}}\, Tgg')U_2 + U_2(A-Kc) - Tgg'U_1 \tag{3.50b}$$

$$U_2(t_0) = 0$$

Proof

First consider the homogeneous part of Eq. (3.49) given by

$$\dot{\underline{a}} = (A-Kc)\underline{a} \qquad (5.8)$$

Assumptions (e) through (h) above, and Theorem 5.4 imply the asymptotic stability of the Kalman-Bucy filter associated with the system DS2. Consequently, the homogeneous system of Eq. (5.8) is asymptotically stable, and futhermore, K is bounded.

Next, we prove that under the assumptions of the theorem, U_1+U_2 is bounded. In Eq. (3.50a) for U_1, F' and (A-Kc) are stability matrices. Therefore, from Lemma 5.2, U_1 is bounded. Furthermore, in Eq. (3.50b) for U_2, $F'-\frac{1}{\sqrt{k}}\,Tgg'$ is also a stability matrix from Theorem 5.6. Thus U_2 is also bounded. Hence U_1+U_2 is bounded and the T-stability of L_2 follows.

Q.E.D.

The above discussion establishes the conditions for the T-stability of the open-loop compensator $W_{co}=M^{-1}*L_2$. Since the impulse response of the overall system is $W*W_{co}$, and since W must be T-stable in accordance with the conditions of Theorem 5.6, the overall system is T-stable.

In summary, we thus conclude that the conditions (a) through (d) of Theorem (5.6), and the conditions (e) through (h) of Theorem (5.7) guarantee the T-stability of the open-loop compensator, and that of the overall system.

5.4 THE CLOSED-LOOP SYSTEM

In this section we discuss the stability properties of the closed-loop compensator and the closed-loop system of Chapter 4. We show below that the stability of the closed-loop system (but not necessarily that of the closed-loop compensator) is a direct consequence of the stability of M^{-1} and L_2.

From Chapter 4, we have

$$y(t) = W * G_c * \{1+F * W * G_c^{-1}\} z(t) \qquad (4.12)$$

where G_c, the impulse response of the closed-loop compensator is given by

$$G_c = [1-M^{-1} * L_2 * F * W]^{-1} * M^{-1} * L_2 \qquad (4.19)$$

Substituting Eq. (4.19) into Eq. (4.12), and for convenience, omitting the asterisk, we obtain

$$y(t) = W(1-M^{-1}L_2FW)^{-1}M^{-1}L_2\{1+FW(1-M^{-1}L_2FW)^{-1}M^{-1}L_2\}^{-1}z(t) \qquad (5.9)$$

Now consider the bracketed portion of Eq. (5.9), which we rewrite below.

$$\{1+FW(1-M^{-1}L_2FW)^{-1}M^{-1}L_2\}^{-1} \qquad (5.10)$$

Expanding $(1-M^{-1}L_2FW)^{-1}$ into a Neumann series [28] Eq. (5.10) is written as

$$\{1+FW(1-M^{-1}L_2FW+M^{-1}L_2FWM^{-1}L_2FW-....)M^{-1}L_2\}^{-1}$$

$$= \{1+FWM^{-1}L_2-FWM^{-1}L_2FWM^{-1}L_2+FWM^{-1}L_2FWM^{-1}L_2FWM^{-1}L_2 ..\}^{-1}$$

$$= \{(1-FWM^{-1}L_2)^{-1}\}^{-1}$$

$$= 1-FWM^{-1}L_2$$

Thus Eq. (5.9) reduces to

$$y(t) = W(1-M^{-1}L_2FW)^{-1}M^{-1}L_2(1-FWM^{-1}L_2)z(t) \qquad (5.11)$$

By expanding $(1-M^{-1}L_2FW)^{-1}$ once again into a Neuman series, it is straightforward to show that

$$(1-M^{-1}L_2FW)^{-1}M^{-1}L_2 = M^{-1}L_2(1-M^{-1}L_2FW)^{-1} \qquad (5.12)$$

Substituting Eq. (5.12) into Eq. (5.11), we obtain

$$y(t) = WM^{-1}L_2(1-M^{-1}L_2FW)^{-1}(1-M^{-1}L_2FW)z(t)$$

$$= W * M^{-1} * L_2 * z(t) \qquad (5.13)$$

The discussion so far, on the closed-loop system leads to the following theorem.

Theorem 5.8

Under the assumptions (a) through (d) of Theorem 5.5 concerning DS1, and the assumption (e) through (h) of Theorem 5.6 concerning DS2, the closed-loop system of Fig. (4.1) is BIBO stable.

Proof

The proof follows immediately from Theorems 5.6, 5.7 and Eq. (5.13).

Q.E.D.

It should be noted that Theorem 5.8 gives conditions for the BlBO stability of the closed-loop system. The T-stability of the system, in addition requires the T-stability of the closed-loop compensator and the feedback sensor (DS3) with impulse response functions G_c and F respectively. From Eq. (4.19)

$$G_c = [1-M^{-1} * L_2 * F * W]^{-1} * M^{-1} * L_2$$

Thus in addition to the conditions of Theorem 5.8 which imply the T-stability of $M^{-1} * L_2$, the T-stability of the closed-loop system requires the T-stability of the operator $[1-M^{-1} * L_2 * F * W]^{-1}$. The additional conditions which must be imposed on DS1, DS2 and DS3 for the T-stability of $[1-M^{-1} * L_2 * F * W]^{-1}$ are left as a topic of future research on this problem.

5.5 SUMMARY AND CONCLUSION

In this chapter we derived conditions for the T-stability of the open-loop compensator and the open-loop system, and the BlBO stability of the closed-loop system. It was shown that the conditions which must be imposed upon the plant (DS1) and the reference system (DS2) for the T-stability of the open-loop system, also guarantee the BlBO stability of the closed-loop system. The T-stability of the closed-loop system requires, in addition, the T-stability of the closed-loop compensator. The T-stability properties of the closed-loop compensator are left as a topic of future research in this area.

CHAPTER 6

CONCLUSIONS AND RECOMMENDATIONS FOR FURTHER RESEARCH

6.1 CONCLUSIONS

The object of this research was to develop a unified theory of optimal tracking (in the mean-square sense) which would admit stationary as well as nonstationary systems. Only single-input/single-output, multi-state systems were considered here. The research, in essence involved the design of compensators which would give the systems under study the desired tracking properties.

The systems in both, an open-loop and a closed-loop configuration, were studied. For each of the configurations, the optimal compensators were realized explicitly in the state-space. The compensator equations contain an unknown parameter (a Lagrange multiplier), whose value is chosen, usually by graphical techniques, to limit the input signal to the plant.

The question of the system stability for the two configurations was also addressed. It was shown that if the Kalman-Bucy filter associated with the input signal, and the given plant are asymptotically stable, the open-loop system is totally stable (T-stable). It was also shown that for the same conditions, the closed-loop system is only bounded-input bounded-output stable. Additional

conditions which must be imposed upon the closed-loop system for the T-stability, need further research.

6.2 RECOMMENDATIONS FOR FURTHER RESEARCH

The theory of stochastic optimal tracking of nonstationary systems is by no means complete at this time. Some related problems and areas which warrant further research are given below.

1. Conditions for only the bounded-input/bounded output stability of the closed-loop system are given here. The question of the T-stability of the closed-loop system warrants further research.
2. Recall from Chapter 5 that a necessary condition for the T-stability of the open-loop system, and the BIBO stability of the closed-loop system, is the asymptotic stability of the plant. What if the plant is unstable to start with. The open-loop compensator is of no direct value here, unless the plant is stabilized first by an auxilliary closed-loop [2]. A possible alternative is the closed-loop compensation scheme along the lines of Youla, Jabr and Bongiorno [5] for the stationary case. Thus if the unstable plant-sensor combination can be stabilized at all, and if Σ is the set of all possible stable and proper compensators which stabilize the system, then the search for the optimal compensator need to be restricted to this

set. An extention of this method to the case of nonstationary systems is nontrivial and warrants considerable research.

3. The case of multi-input/multi-output system and colored noise should be investigated. Extention of all of the above work, completed or suggested, for the discrete time systems is of great practical value.

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